THE SYNTHESIS OF THE ELEMENTS FROM HYDROGEN *

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Summary

Stars that have exhausted their supply of hydrogen in regions where thermonuclear reactions are important enter a collapsing phase. If the mass of the star exceeds Chandrasekhar's limit collapse will continue until rotational instability occurs. Rotational instability enables the star to throw material off to infinity. This process continues until the mass of the remaining stellar nucleus becomes of the order of, or less than Chandrasekhar's limit. The nucleus can then attain a white dwarf equilibrium state.

The temperature generated at the centre of a collapsing star is considered and it is shown that values sufficiently high for statistical equilibrium to exist between the elements must occur. The relative abundances of the elements can then be worked out from the equations of statistical mechanics. These equations are considered in detail and it is shown that a roughly uniform abundance of the elements over the whole of the periodic table can be obtained. The process of rotational instability enables the heavy elements built up in collapsing stars to be distributed in interstellar space.

The results arising from the discussion of the formation of heavy elements lead to a natural explanation of the difference between novae and supernovae.

1. *Introduction*

1.1. *Preliminary remarks.*—The rate at which nuclear re ctions take place in a thermodynamic assembly of material particles and radiation is negligible at ordinary temperatures, except in the case of radioactive nuclei. But as the temperature T increases, the thermal energy of the particles increases until for temperatures in the neighbourhood of 5×10^9 °C. appreciable penetration of the Coulomb barriers of the nuclei by protons and α-particles occurs. Thus, provided the density of the particles is high enough for collisions to be sufficiently frequent the rate of nuclear reactions becomes important at very high temperatures. Indeed for sufficiently large ρ, T the interchanges between nuclei of different atomic weights and nuclear charges become so rapid that statistical equilibrium is set up between the nuclei. The relative abundances of the various elements can then be calculated in terms of ρ, T from the equations of statistical mechanics. Moreover the relative abundances so calculated are independent of the initial composition of the assembly. Thus if we start with an assembly of hydrogen at low temperature and ρ, T are increased until statistical equilibrium occurs the relative abundance will be independent of starting with hydrogen and would have been the same if, for example, we had started with an assembly of helium. The work of Section 5 shows that statistical equilibrium will be set up in a time interval of less than 100 seconds when $\rho = 10^7$ gm. per cm.^{3} and $T = 4 \times 10^9$ °C. This result is very sensitive to the value of T but not to the value of ρ.

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No attempt will be made in the present paper to discuss non-statistical conditions of synthesis of the elements from hydrogen. It is, however, of interest to consider whether this restriction will affect the generality of the conclusions reached. In the writer's opinion no such restriction can arise. There are two arguments that support this point of view. Firstly in order to synthesize the heavy elements on the scale required to explain the occurrence of the heavy elements in the Earth it is necessary that a large number of interchanges between atomic nuclei shall take place, and that these interchanges take place over the whole range of atomic weights. This, however, is essentially the condition required for statistical equilibrium. Stated otherwise, the present argument means that if an appreciable fraction of the hydrogen is to be converted into heavy elements then statistical equilibrium must occur (at any rate approximately) since the number of nuclear reactions necessary to bring about the synthesis must be comparable with, or greater than, the number of particles involved (if only a minute fraction of the hydrogen were converted into heavy elements then non-statistical processes such as occur in laboratory nuclear physics could take place). In addition to this general argument there is an important special argument. In order to synthesize the naturally radioactive elements it is necessary to consider reactions that are the reverse of the radioactive disintegrations. Under statistical equilibrium no special difficulty arises, but under non-statistical conditions it seems impossible to proceed (in a reverse direction) along the radioactive series beyond such extremely short-lived nuclei as *Ra C′* and *Th C′*.

The above remarks suggest the problem of finding values of ρ, T such that the statistical equations give the relative abundances of the elements observed in nature. The phrase "in nature" requires some discussion. The information available concerning the distribution of the elements may be summarized as follows:—

(1) It is known that a large quantity of material occurs both in the Galaxy and in the extragalactic nebulae in the form of a diffuse interstellar gas. Although the amount of this material is not yet accurately determined, there is important observational evidence in favour of the view that the total mass of interstellar material in a nebula exceeds the combined masses of the stars by an appreciable factor. The interstellar material is believed to consist very largely of hydrogen, the quantity of heavy elements amounting to only a few per cent. at most. The heavy elements are present partly in gaseous form and partly as interstellar dust.

(2) As a result of a fairly thorough analysis of the structure of the stars it is now known that a large fraction of the combined masses of the stars is due to hydrogen. In certain stars hydrogen contributes at least 70 per cent., whereas at the other end of the scale stars are known to occur (see Section 2) in which hydrogen is almost entirely absent. This diversity in hydrogen content is due to the interplay of two processes with opposing tendencies:

(i) The generation of energy within a star due to thermonuclear reactions has the net effect of synthesizing helium from hydrogen (in stars of mass less than about half the solar mass $M_\odot$ the helium is commonly believed to be synthesized directly from the hydrogen, whereas in stars of mass greater than about $M_\odot/2$ the synthesis occurs through the carbon-nitrogen cycle). Thus the effect of nuclear reactions *within a normal star* is to increase the helium concentration at the expense of the hydrogen (the distinction between a normal star and a collapsing star is discussed in detail in Section 2). The synthesis from hydrogen of elements heavier than helium is negligible in normal stars * in which the central temperatures never appreciably exceed 5×10^7 °C.

(ii) The loss of hydrogen by synthesis of helium is offset in some stars by rapid accretion of interstellar hydrogen.†

Accordingly we form the following picture of the development of a star during the normal phase of its history.‡ The star condenses from the diffuse interstellar material § and has initially the same composition as the interstellar material. As a result of thermonuclear reactions hydrogen is converted into helium. In some stars the loss of hydrogen is offset by a rapid accretion of further quantities of interstellar material.

(3) Additional data concerning the composition of the stars is obtained from the study of stellar spectra. Although the identification of elements over the whole periodic table is not yet completed, the results so far obtained provide strong confirmation of the view that all elements known on the Earth are also present in the stars. It has been estimated by Russell || that in the surface layers of the stars hydrogen is at least a thousand times as abundant as all the metals combined.

(4) By far the most detailed evidence concerning the distribution of heavy elements comes from the direct study of the contents of the earth's crust.¶ The distribution so obtained differs markedly from the astrophysical evidence discussed above in that only a small quantity of hydrogen and an almost negligible amount of helium is present on the Earth. The most common elements in the crust are silicon, magnesium, and oxygen.

In the present paper we are not so much concerned with the abundance of isolated elements as with suitably averaged values. We may group the elements into a number of ranges of atomic weight, e. g. the ranges 1 to 14 inclusive, 15 to 24 inclusive etc., and average the abundance of all the elements falling within each group. When this is done an averaged abundance is obtained for each range of atomic weight. These averaged abundances rarely fall below one part in a million (uranium forms about one part in 10^5 and lead about five parts in 10^6, so that this statement holds up to the heaviest elements in the periodic table). Now although a difference in abundance between two elements by a factor of 10^3 may be of very great economic importance (one of the elements being regarded as "common" and the other as "rare") such a difference is unimportant from the standpoint of the present paper since factors arise in the present discussion that may be as large as 10^{200}. From the work given below it will appear that a remarkable feature of the survey of elements present in the Earth's crust is the very small variation in abundance that occurs over the whole range of atomic weights.

The above remarks concerning the uniformity of distribution of the elements in the Earth's crust are strengthened by noticing that the heaviest elements would

* H. A. Bethe, *Phys. Rev.*, **55**, 434, 1939.
† F. Hoyle and R. A. Lyttleton, *M.N.*, **101**, 227, 1941.
‡ See Section 2.
§ F. Hoyle, *M.N.*, **105**, 302, 1945.
|| H. N. Russell, *Ap. J.*, **78**, 239, 1933.
¶ V. M. Goldschmidt, *Skr. norske Vidensk Akad.*, 1937.

sink towards the centre when the Earth was in a liquid state. Thus the proportion of elements with atomic weights greater than 40 in the Earth as a whole may well be considerably higher than is indicated by a study of the crust alone. Results obtained from observations of the propagation of sound waves through the body of the Earth confirm this statement. It is found* that the density of material near the centre of the Earth is close to twelve. Allowing for increase of density arising from the high pressure in the inner regions of the Earth it has been estimated that such material would have a density of about eight under normal laboratory conditions. This means that the central regions of the Earth are occupied by material with a density close to that of iron or copper. It is usually inferred from this result that the core of the Earth is occupied by elements with atomic weights lying between 50 and 70. The core may however contain ten per cent. by mass of the heaviest elements without any disagreement with observation arising.

It will be seen from the above remarks that the evidence available concerning the distribution of the elements in the universe is very fragmentary. Nevertheless the astrophysical data are sufficient to bring out the important result that hydrogen is by far the most abundant element. This conclusion gives strong support to the view that it should be possible to trace back the history of the universe to a state in which hydrogen is the only element present. That is, it must be possible to show how the various elements have been synthesized from hydrogen. The present paper attempts to show how such a synthesis takes place. Stated more explicitly we suppose:

(i) Initially the only element present in the universe is hydrogen.

(ii) Helium is synthesized by thermonuclear reactions taking place in "normal" stars.

(iii) A further process occurs that synthesizes higher elements from hydrogen and helium. The elements produced are regarded as having a distribution similar to that found on the Earth.

The object of the present paper is to discuss (iii).

1.2. *Relation with previous investigations.*—The equations of statistical equilibrium between atomic nuclei have been discussed by a number of authors.† The most recent work is by Chandrasekhar and Henrich (referred to below as the C-H theory). The C-H theory deals with the relative abundance of the elements at temperatures between 8×10^9 and 6×10^9 °C. and at densities of order 10^7 gm. per cm.^{3}. Under these conditions only about one part in a million by mass is converted into elements of atomic weight greater than that of helium. The relative abundances of the elements are well reproduced in the range between *C* and *Cl*, but beyond atomic weight 40 the abundances given by the C-H theory become quite negligible. Thus the C-H theory fails to account for the existence of elements beyond argon. One of the main features of the present work is that the required abundances will be obtained over the whole of the periodic table. The chief points of difference between the present and previous discussion may be summarized as follows:—

(1) Account is taken of the degeneracy of the electrons at high densities.

* H. Jeffreys, The Earth, p. 220, Cambridge, 1929; *M.N.*, *Geophys. Suppl.* **4**, 62, 1937.

† T. E. Sterne, *M.N.*, **93**, 736, 1933; R. H. Fowler, Statistical Mechanics, p. 655, Cambridge, 1936; S. Chandrasekhar and L. R. Henrich, *Ap. J.*, **95**, 288, 1942.

(2) A large fraction of the material is converted into elements heavier than helium (as opposed to one part in a million in the C-H theory).

(3) The inclusion of electron degeneracy enables the required relative abundances to be obtained over the whole range of the periodic table without the temperature of the assembly being appreciably greater than 10^{10} °C.

In addition to these technical points the present work differs from the C-H theory on a number of questions of principle:

(*a*) It is not sufficient to find the values of ρ, T required for an assembly to contain the proper relative abundances of the elements. It is also necessary that ρ, T can be converted to low density and temperature (such as occurs on the Earth or in interstellar space) *without the relative abundances being changed*. This means that the mixture must remain frozen during some cooling-expansion process, for if statistical equilibrium were maintained throughout the process the relative abundances would undergo important changes.

(*b*) It is necessary to find the *place in the universe* where the required values of ρ, T occur.

In addition to previous investigations dealing with the synthesis of elements there are also a number of papers concerned with the properties of matter at high temperatures and densities that the writer has read with interest.*

1.3. *Procedure to be followed.*—The chief aims of the present paper may be summarized briefly:—

(1) To find values of ρ, T such that the equations of statistical equilibrium give the required abundances of the elements over the whole range of the periodic table.

(2) To show that a cooling-expansion process can be found such that the values of ρ, T satisfying (1) can be reduced to low density and temperature without the composition of the mixture being changed.

(3) To find a place in the universe where the values of ρ, T satisfying (1) must occur and to explain in a satisfactory way the origin of the cooling-expansion process.

There are two ways of discussing this programme. The indirect way is to construct the equations of statistical equilibrium and to search by trial and error until values of ρ, T are obtained that satisfy (1). The next step would be to use these values of ρ, T in a search for processes satisfying (2), (3) and (4). The second and more direct way is to discuss (3) and to find what values of ρ, T satisfy (2) also. Then having arrived at a pair of values of ρ, T (or possibly at more than one pair of values) we can use (1) and (2) as consistency checks on the theory. The latter procedure will be adopted in the following work. It is shown below that consideration of (2) and (3) leads to two possible ranges of values of ρ, T. One of these ranges yields the abundance of the elements of atomic weight up to about 80 (this is material described as case (1) in Section 8), whilst the other case yields the required abundances over the remainder of the periodic table (this material is described in Section 8 as case (2)).

An important by-product of the present investigation is a natural explanation of the difference between novae and supernovae. The theory of supernovae here given is similar in a number of respects to the neutrino-emission theory

* W. Baade and F. Zwicky, *Phys. Rev.*, **45**, 138, 1934 and **46**, 76, 1934; L. Landau, *Nature*, **141**, 333, 1938; F. Zwicky, *Phys. Rev.*, **55**, 726, 1939; F. Cernushi, *Phys. Rev.*, **56**, 450, 1939.

given by Gamow and Schoenberg.* The chief point of difference is that whereas in the Gamow-Schoenberg theory removal of energy occurs only through neutrino emission, in the present theory a more rapid rate of removal of energy arises from nuclear transformations.

2. *Some General Properties of Stars*

2.1. *Normal Stars.*—In the present section we shall consider stars in which nuclear transformations do *not* abstract thermal energy from stellar material (as will be seen in later sections important cases can occur where the absorption of energy by nuclear reactions becomes the dominant process in determining the evolution of a star). Under these conditions the only important way in which a star loses energy is by radiation from the surface. It follows from this, as will be proved at the end of the present subsection, that the star must satisfy the hydrostatic equation

$$\mathfrak{R}\frac{d}{dr}\left(\frac{\rho T}{\mu\beta}\right) = -\frac{GM(r)\rho}{r^2} \tag{1}$$

to a very high degree of approximation, where $\mathfrak{R}$ is the gas constant; μ, ρ, T are the mean molecular weight, density and temperature of the stellar material; β is the ratio of the gas pressure $\mathfrak{R}\rho T/\mu$ to the total pressure $\mathfrak{R}\rho T/\mu + aT^4/3$, which includes the pressure $aT^4/3$ due to radiation, where a is Stefan's constant; G is the gravitational constant, r is a radial coordinate that measures distance from the centre of the star, and $M(r)$ is the mass of the star contained in a sphere of radius r.

Now when a star satisfies (1) either accurately or approximately the central temperature T_c cannot differ by a factor of much more than two from the value †

$$\frac{G\mu\beta_c M}{3\mathfrak{R}R}, \tag{2}$$

where M is the total mass of the star, β_c is the value of β at the centre, and R is the radius of the star (more accurately R should be taken such that $M(R) \simeq M/2$, because it is possible for a star to possess an outer envelope of large radius but containing only a small fraction of the mass if μ is appreciably less in the envelope than in the interior ‡). The temperature of the surface of the star is small compared with (2) and there must be an outward temperature gradient of order

$$\frac{G\mu\beta_c M}{3\mathfrak{R}R^2}$$

in the star. This temperature gradient will result in there being an outward flux of energy from inside the star to the surface. The magnitude of this energy flux follows immediately from the well known equation of radiative equilibrium and is of order

$$\frac{16\pi caR}{3\kappa\bar{\rho}^2(1-X^2)}\left(\frac{G\mu\beta_c M}{3\mathfrak{R}R}\right)^{7\cdot 5}, \tag{3}$$

where κ is the coefficient of opacity, c is the velocity of light, X is the fraction

* G. Gamow and M. Schoenberg, *Phys. Rev.*, **59**, 539, 1941; G. Gamow, *Phys. Rev.*, **59**, 617, 1941 and **65**, 20, 1944.

† A. S. Eddington, I.C.S., p. 92, Cambridge, 1930.

‡ F. Hoyle and R. A. Lyttleton, *M.N.*, **102**, 218, 1942.

by mass of hydrogen in the star, and $\bar{\rho}$ the mean density of material in the star is given by

$$\frac{4}{3}\pi R^3\bar{\rho}=M. \tag{4}$$

Substituting for $\bar{\rho}$ in (3) leads to the following expression for the energy flux:

$$\frac{3ca}{\pi\kappa(1-X^2)}\cdot\left(\frac{G\mu\beta_c}{3\mathfrak{R}}\right)^{7\cdot5}.M^{5\cdot5}R^{-0\cdot5}. \tag{5}$$

The expression (5) not only gives the outward flux of energy in the star but also gives the rate of radiation from the surface. Thus, if the rate of radiation from the surface is greater than (5), the surface temperature must fall, since the loss of energy at the surface cannot be compensated by the flux of energy from inside the star. Similarly, if the radiation from the surface is less than (5), the surface temperature must rise, since the supply of energy from inside the star exceeds the loss by radiation.

If thermonuclear reactions taking place in the star generate sufficient energy to compensate exactly for the loss of energy at the surface, then the star remains in equilibrium. If the generation of energy exceeds the rate of loss then the star will expand. As will be seen from (2) the temperature in the energy-producing regions falls with the expansion, and this has the effect of rapidly reducing the rate of production of energy (the carbon nitrogen cycle gives an energy production approximately proportional to ρ^2T^{18}, whilst direct synthesis helium from hydrogen gives an energy generation approximately proportional to $\rho^2T^{4\cdot5}$). Thus the star expands until the energy production falls to a value equal to the loss at the surface. If, on the other hand, the generation of energy due to the synthesis of helium from hydrogen is less than the loss at the surface, then the star will contract and the temperature will rise in accordance with (2). Thus in this case, provided the quantity of hydrogen present in the energy-producing regions is not negligible, the star will contract until energy production becomes equal to the loss at the surface (the discussion given in 2.2 shows that the present proviso is very important). We may conclude that stars containing an appreciable amount of hydrogen will adjust their dimensions (and hence their central temperatures) in such a way that the energy generation is just sufficient to balance the loss at the surface, which is given approximately by (5).

Now the generation of energy by the carbon nitrogen cycle for example (similar remarks apply to direct synthesis of helium from hydrogen) can be written in the approximate form *

$$4\pi E\int_0^R \rho^2T^\eta r^2\,dr, \tag{6}$$

provided the central temperature is not too far from the equilibrium value, where η varies nearly as $M^{-1/12}$ and is about 18 for the Sun, and E is an energy production constant. The integral (6) can be written as

$$\frac{4}{3}\pi E\lambda R^3\bar{\rho}^2T_c^\eta,$$

where λ is an averaging constant that has effectively the same value from one star

* F. Hoyle and R. A. Lyttleton, *M.N.*, **102**, 177, 1942.

to another. Substituting for $\bar{\rho}$ from (4) and for T_c from (2) gives an energy production

$$\frac{3E\lambda}{4\pi}\left(\frac{G\mu\beta_c}{3\mathfrak{R}}\right)^{\eta}\frac{M^{\eta+2}}{R^{\eta+3}}. \tag{7}$$

For equilibrium conditions the expressions (5) and (7) are equal and R is given approximately by

$$\left\{\frac{E\lambda\kappa(1-X^2)}{4ac}\right\}^{\frac{2}{2\eta+5}}\cdot\left(\frac{G\mu\beta_c}{3\mathfrak{R}}\right)^{\frac{2\eta-15}{2\eta+5}}\cdot M^{\frac{2\eta-7}{2\eta+5}}. \tag{8}$$

Thus, since it can be shown that β_c can be expressed in terms of μ and M, it follows that R can be calculated* explicitly in terms of μ and M. The value of R so determined may be substituted in (5) to give the rate of emission L at the surface in terms of μ and M. Thus *both* R and L can be calculated explicitly in terms of μ and M. These remarks refer to stars in equilibrium. Such stars are *normal* in the sense used in the Introduction.

In a refined discussion of the structure of stars the approximate expressions (5) and (7) have to be replaced by accurate values (the expressions (5) and (7) are approximate in the numerical constants multiplying them and not in the dependence on M, R and μ).† For the present purposes the above discussion is preferable because a number of points are brought out that tend to be obscured in a more detailed theory. Moreover the present investigation shows what results must of necessity turn up in a thorough-going analysis. For example, it is seen from (8) that $R\propto E^{2/(2\eta+5)}$ and from (5) that $L\propto E^{-1/(2\eta+5)}$, so that both R and L are very insensitive to the value of the energy production constant. Furthermore we see that this result is inherent in the problem and cannot be removed by more detailed calculations.

Apart from a numerical factor the expression (5) is identical with a well-known formula due to Eddington.

At this stage we may dispose of the asumption made at the beginning of the present subsection. That is, we have to show that a collapsing star which *only loses energy through radiation at the surface* must satisfy (1) to a high degree of approximation. The total thermal energy contributed by the material of a star is of order $\mathfrak{R}MT_c$. Now the star cannot contract appreciably until an amount of energy has been radiated at the surface that is of the same order as $\mathfrak{R}MT_c$. Thus the time required for appreciable contraction to occur is of order $\mathfrak{R}MT_c/L$. To obtain an idea of the magnitude of this quantity put $M=M_\odot=2\times10^{33}$ gm., $T_c=2\times10^{7}$ °C., $L=L_\odot=4\times10^{33}$ ergs per sec. Then the time required for appreciable contraction to occur is of order $2\cdot8\times10^{7}$ years. This time may be compared with the time that would be required for collapse if (1) were not satisfied to a good approximation. The latter time is easily seen to be of order $\sqrt{(R^3/GM)}$. For $R=R_\odot=7\times10^{10}$ cm. and $M=M_\odot$, $\sqrt{(R^3/GM)}$ is about $1\cdot6\times10^{3}$ sec. Although for different M, T_c, R the numerical values will differ from those given here it is clear that in all cases

$$\frac{\mathfrak{R}MT_c}{L}\gg\sqrt{\left(\frac{R^3}{GM}\right)}.$$

* F. Hoyle and R. A. Lyttleton, *loc. cit.*
† F. Hoyle and R. A. Lyttleton, *loc. cit.*

This proves the required result that (1) must be satisfied to a good approximation when the only loss of energy is by radiation at the surface.

2.2. *Collapsing Stars.*—In this subsection we consider a collapsing star in which the rate of generation of energy is less than the rate of radiation from the surface. Two different cases may be distinguished:

(1) The star contains an appreciable fraction by mass of hydrogen but the internal temperature is too low to give sufficient energy generation for equilibrium. This case occurs during the condensation of a star from interstellar material, and the collapse continues until the temperature rises to a sufficiently high value for equilibrium to occur.

(2) The generation of energy by synthesis of helium from hydrogen may cease, due to the exhaustion of the supply of hydrogen in the energy-producing regions (hydrogen may still be present in the atmosphere). This will occur in a period of about 10^8 years (compared with an age of the universe of about 10^{10} years) for stars of large mass and luminosity unless the hydrogen is replenished by accretion of interstellar hydrogen. Thus, if accretion ceases to be important * for such a star, the supply of hydrogen will become exhausted in a period of time that is short compared with the age of the universe. The star will then enter a period of contraction. As the radius decreases the central temperature will increase in accordance with (2). The rising temperature will remove the last vestiges of hydrogen present in the inner parts of the star and the star if sufficiently massive must collapse until rotational instability occurs.† The condition on the mass is that it must appreciably ‡ exceed Chandrasekhar's limit $5{\cdot}75 M_{\odot}/\mu_e^2$. Sufficient material is thrown off to infinity during the instability process for the mass of the remaining stellar nucleus to be reduced to a value of the order of, or less than, Chandrasekhar's limit.

From the point of view of the present paper two different cases may be distinguished. These two cases may be examined by discussing the instability process in further detail.

(i) The temperature rises sufficiently high for nuclear reactions to become important *before* the onset of rotational instability (a temperature of about 4×10^9 °C. is required).

(ii) A state of rotational instability is reached before nuclear reactions become important.

Owing to the conservation of angular momentum, any initial rotation possessed by a star increases as the star contracts. During contraction the rotational velocity is proportional to the reciprocal of the radius of the star. Thus the centrifugal force acting on the stellar material increases as the inverse cube of the radius. It follows that, since the gravitational force increases only as the inverse square of the radius, the ratio of centrifugal force to gravitational force must increase inversely as the radius. When the centrifugal force becomes comparable with the gravitational force, instability occurs. If we regard equality

* F. Hoyle, *M.N.*, **105**, 363, 1945.

† F. Hoyle, *M.N.*, in course of publication.

‡ The quantity μ_e is defined somewhat differently from μ. Thus whereas μ is given by the product of the mass of the hydrogen atom and the number of particles per gram of material, the quality μ_e is defined as the product of the mass of the hydrogen atom and the number of free electrons per gram of material. For the stars considered in the present paper μ_e is always close to 2.

of these two forces as an approximate criterion for rotational instability, then we can estimate the degree of contraction required to produce instability for various values of the initial equatorial rotational velocity. For example, consider a star of mass $10M_\odot$ that exhausts its supply of hydrogen and enters the period of contraction discussed above. Before contraction begins, the values of ρ, T at the centre of the star would be about 3 gm. per cm.^{3} and $3{\cdot}7\times10^7$ °C., whilst the radius would be about $3{\cdot}5\times10^{11}$ cm. To a sufficient approximation we may suppose that the star contracts through a series of homologous configurations, so that the central values of ρ, T are proportional to the inverse cube and the inverse first power respectively of the radius. The following table gives the various quantities that are of interest for a set of values of the initial rotational velocity:—

Table I

The Effect of the Initial Rotational Velocity on Collapsing Stars

Mass $=10M_\odot$. Initial radius $=3{\cdot}5\times10^{11}$ cm.

Initial rotational velocity (km. per sec.).	1	5	10	15	20	40	100
Factor by which radius must be reduced to make centrifugal and gravitational forces equal.	$3{\cdot}82\times10^5$	$1{\cdot}53\times10^4$	$3{\cdot}82\times10^3$	$1{\cdot}70\times10^3$	$9{\cdot}52\times10^2$	$2{\cdot}38\times10^2$	$3{\cdot}82\times10$
Initial central temperature in °C.	$3{\cdot}5\times10^7$	$3{\cdot}5\times10^7$	$3{\cdot}5\times10^7$	$3{\cdot}5\times10^7$	$3{\cdot}5\times10^7$	$3{\cdot}5\times10^7$	$3{\cdot}5\times10^7$
Central temperature at time of rotational instability in °C.	$>4\times10^9$	$>4\times10^9$	$>4\times10^9$	$>4\times10^9$	$>4\times10^9$	$>4\times10^9$	$1{\cdot}34\times10^9$
Initial central density in gm. per cm.^{3}.	3	3	3	3	3	3	3
Central density at onset of rotational instability in gm. per cm.^{3}.	$1{\cdot}67\times10^{17}$	$1{\cdot}07\times10^{13}$	$1{\cdot}67\times10^{11}$	$1{\cdot}47\times10^{10}$	$2{\cdot}59\times10^9$	$4{\cdot}07\times10^7$	$1{\cdot}67\times10^5$

The values given in Table I are necessarily approximate (in particular relativity effects are neglected in the first set of values), but even so they show clearly that unless the initial rotational velocity of the star is of the order of, or greater than, 100 km. per sec. the temperature will rise to 4×10^9 °C. before rotational instability occurs. Now at a temperature of 4×10^9 °C. nuclear reactions become important and we have case (i). Indeed it follows from Table I that, if the initial rotational velocity is of the order of, or less than, 40 km. per sec., we have case (i), whereas, if the initial rotational velocity is of the order of, or greater than, 100 km. per sec., we have case (ii). Thus, since the initial rotational velocity will vary from one star to another, it follows that some stars will satisfy case (i) and others case (ii). In the present paper we are only concerned with stars in which the initial rotational velocity is small enough for case (i) to apply. The range of values of the initial rotational velocity used in the following work is roughly 10 to 40 km. per sec.

The large densities appearing in the above table suggest that electron degeneracy may arise during the contraction of the star, and that the contribution of the pressure of the degenerate electrons may become important. As will be seen in a later section, this is the case, but only *after* nuclear reactions have become important. Before nuclear reactions become important, the density increases at the same rate as the cube of the temperature, so that the ratio of gas pressure to radiation pressure remains constant during this phase of the contraction. Now, as Chandrasekhar has shown*, pressure degeneracy does not become important unless β_c is greater than a value close to 0·908. From the known relation† between β_c and M, it can be shown that β_c will not exceed 0·9 unless $M\mu_e^2/M_\odot < 5{\cdot}75$. In the following discussion we shall only consider stars satisfying $M\mu_e^2/M_\odot > 5{\cdot}75$, so that degeneracy does not arise before nuclear reactions become important. If a star undergoes appreciable contraction after the temperature has risen to 4×10^9 °C., then degeneracy will always occur. This is due to nuclear reactions which prevent the temperature from rising appreciably above 10^{10} °C. At this stage β_c increases, because the gas pressure continues to increase on account of the increasing density but radiation pressure remains effectively constant. If sufficient contraction occurs, β_c will rise above 0·908 and the stellar material will become degenerate.

2.3.—The remarks of 2.2 provide a suitable basis for a theory of the synthesis of the elements. It has been shown that the temperatures arising in collapsing stars that have exhausted their supply of hydrogen become sufficiently high for statistical equilibrium between atomic nuclei to occur provided

(1) The mass of the star appreciably exceeds Chandrasekhar's limit.

(2) The initial rotational velocity before collapse begins is less than about 40 km. per sec.

Thus our first object of finding a place in the universe where the elements may be synthesized has been achieved. Furthermore, a cooling-expansion process of the type discussed in the Introduction is also available in the process of rotational instability. This serves not only to reduce the density and temperature but to distribute the elements in interstellar space.

In view of the importance of the processes discussed in 2.2 it is desirable to seek independent confirmation of these results. It is therefore fortunate that observational evidence can be cited in support of the starting-point of the present investigations. Thus what appears to be an excellent example of the break up of a rotationally unstable star has recently been investigated in detail by Baade and Minkowski.‡ In this particular case it is possible to observe directly the nebulous material that was thrown off by the supernova of 1054 A.D. The mass of this material is found to be of order 15 $M_\odot$ so that the star must have thrown off an appreciable proportion of its original mass. The estimate given by Minkowski for the radius of the remaining stellar nucleus confirms that the material in the nucleus is very dense, and thereby provides support for the view that the outburst occurred in a collapsing star whose supply of hydrogen had become exhausted. Thus, if we take the mass in the nucleus as $M_\odot$, which is probably an underestimate, together with Minkowski's estimate of about 10^9 cm. for the radius, the present mean density is about 5×10^5 gm. per cm.^{3}. Allowing for a central condensation of about 20 (the central density in

* S. Chandrasekhar, Introduction to the Study of Stellar Structure, p. 434, Chicago, 1938.

† F. Hoyle and R. A. Lyttleton, *loc. cit.*

‡ W. Baade, *Ap. J.*, 96, 188, 1942 ; R. Minkowski, *Ap. J.*, 96, 199, 1942.

a star is about 20 times the mean density), this gives a present central density in the nucleus of about 10^7 gm. per cm.^{3}. We may note further that the central density before outburst would be unlikely to be less than 10^7 gm. per cm.^{3}. Thus, taking 16 $M_\odot$ as a rough estimate of the initial mass, it follows that the radius at outburst could not be much greater than $2{\cdot}5 \times 10^9$ cm. If this value of the radius is used in (2), together with $\mu\beta_c = 1$, an estimate of $3{\cdot}4 \times 10^9$ °C. is obtained for the central temperature before outburst. This value is just of the order that is of interest in the present problem. Accordingly it is seen that there is not only theoretical evidence but also observational data in favour of the model used in the following sections of the present paper.

3. *The Equations of Statistical Equilibrium*

The next step is to construct the equations of statistical equilibrium between atomic nuclei (these equations applying at temperatures exceeding 4×10^9 °C.). Write n_P, n_N, n_A^Z for the number of free protons per cm.^{3}, the number of free neutrons per cm.^{3}, and the number of atomic nuclei with atomic weight A and charge Z per cm.^{3}, respectively. Then under conditions of statistical equilibrium n_A^Z can be expressed in terms of the three quantities n_P, n_N and T. These equations are well known and will be quoted here.* In Fowler's notation the required equations are

$$\left.\begin{aligned} Vn_P &= \xi \frac{\partial}{\partial \xi}\left\{\sum_q \ln\left(1 + \xi e^{-\epsilon_q/kT}\right)\right\}, \\ Vn_N &= \zeta \frac{\partial}{\partial \zeta}\left\{\sum_r \ln\left(1 + \zeta e^{-\epsilon_r/kT}\right)\right\}, \\ Vn_A^Z &= \chi \frac{\partial}{\partial \chi}\left\{\sum_s \pm \ln\left(1 \pm \chi e^{-\epsilon_s/kT}\right)\right\}, \end{aligned}\right\} \qquad (9)$$

where V is the volume occupied by the assembly and ϵ_q, ϵ_r, ϵ_s represent the energy states (energy of translation together with internal excitation) of the proton, neutron and nucleus of atomic weight A and charge Z, respectively. The $\pm$ sign depends on A; if A is odd the nuclei obey Fermi-Dirac statistics and the plus sign must be taken, whilst if A is even the nuclei obey Einstein-Bose statistics and the minus sign must be used. The parameters ξ, ζ, χ appear in using the theorem of steepest descents and they satisfy the relations

$$\left.\begin{aligned} \chi &= \xi^Z \zeta^{(A-Z)} \exp\left(-Q_A^Z/kT\right), \\ Q_A^Z &= c^2\{m_A^Z - Zm_P - (A-Z)m_N\}, \end{aligned}\right\} \qquad (10)$$

m_P, m_N, m_A^Z being the masses of the proton, neutron and nucleus of weight A and charge Z, respectively. There is a set of equations of the form (9) and (10) for each pair of values of A, Z.

The above equations can be simplified in two ways. First, to the degree of accuracy aimed at in the present paper the contribution to ϵ_s of the excited states of the nuclei can be neglected (the errors arising from this step are not greater than a factor of about 10^2 in the relative abundances of the nuclei. This is small compared with the factors that arise from other considerations). Second,

* R. H. Fowler, *op. cit.*, p. 655. The symbol " ln " denotes the logarithm to the base e, and the symbol " log " will be used to denote logarithms to the base 10.

even at the highest densities discussed below ($\simeq 10^{11}$ gm. per cm.^{3}), the parameters ξ, ζ, χ are sufficiently small compared with unity for it to be possible to use "classical" statistics for the protons, neutrons and heavy nuclei. These simplifications enable the sums in (9) to be written as *

$$\left.\begin{aligned} \sum_q \ln(1+\xi e^{-\epsilon_q/kT}) &= \xi V(2\pi m_P kT)^{\frac{3}{2}}/h^3, \\ \sum_r \ln(1+\zeta e^{-\epsilon_r/kT}) &= \zeta V(2\pi m_N kT)^{\frac{3}{2}}/h^3, \\ \pm\sum_s \ln(1\pm\chi e^{-\epsilon_s/kT}) &= \chi V(2\pi m_A^Z kT)^{\frac{3}{2}}/h^3. \end{aligned}\right\} \qquad (11)$$

Equations (11) together with (10) give

$$\ln n_A^Z = Z\ln n_P + (A-Z)\ln n_N + (A-1)\ln\{h^3/(2\pi kT)^{\frac{3}{2}}\} - \frac{Q_A^Z}{kT} - \frac{3}{2}Z\ln m_P - \frac{3(A-Z)}{2}\ln m_N + \frac{3}{2}\ln m_A^Z. \qquad (12)$$

This equation has been expressed in a convenient numerical form by Chandrasekhar and Henrich †, who use 10^9 °C. as the unit of T, and the millimass unit ($10^{-3}\ m_P$) as the unit of Q_A^Z. With these units, and putting $m_P = m_N$, equation (12) becomes

$$\log n_A^Z = 34{\cdot}08 + \frac{3}{2}\log T + \frac{3}{2}\log A + \frac{4{\cdot}73}{T}\,.\,Q_A^Z + A\left(\log n_N - 34{\cdot}08 - \frac{3}{2}\log T\right) - Z\log\left(\frac{n_N}{n_P}\right). \qquad (13)$$

Equation (13) gives one of the basic equations of the present problem. The remaining equations are rather less straightforward in their derivation and will be treated in greater detail. A relation can be derived between n_N/n_P and n_e, where n_e is the electron density. This relation is due to interchange between an electron and a proton on the one hand and a neutron on the other hand. The interchange arises from β-processes taking place among the nuclei. The neutrinos and antineutrinos (according to the usual formulation of the properties of these particles) emitted in the β-processes will escape from the star and will remove energy in the manner discussed by Gamow and Schoenberg.‡ Provided the interchanges are sufficiently frequent (we shall assume this to be the case), statistical equilibrium will be set up, which will be similar in character to the case of a simple dissociating gas. In the latter problem we have atoms of type A_1 combining with atoms of type A_2 to form molecules $A_1 A_2$, together with the reverse reaction in which $A_1 A_2$ dissociates into the separate atoms, A_1 and A_2. This can be represented symbolically as

$$A_1 + A_2 \rightleftarrows A_1 A_2.$$

The solution of this problem is known § and can be taken over to the proton, neutron, electron case by regarding the neutron as the "molecule" and the proton and electron as the "atoms". The equations determining the equilibrium are

* R. H. Fowler, *op. cit.*, p. 655.
† Chandrasekhar and Henrich, *op. cit.*
‡ Gamow and Schoenberg, *loc. cit.*
§ Fowler, *op. cit.*, p. 157.

the first two equations of (9) together with

$$\left.\begin{aligned} Vn_e &= \lambda \frac{\partial}{\partial\lambda}\left\{\sum_t \ln\left(1+\lambda e^{-\epsilon_t/kt}\right)\right\}, \\ \zeta &= \lambda\xi \exp(-\mathcal{J}/kT), \\ \mathcal{J} &= c^2(m_N - m_P - m_e), \end{aligned}\right\} \tag{14}$$

where ϵ_t represents the energy levels of the electron and m_e is the electron mass. Putting ζ, ξ equal to the values given by (9) and (11) gives

$$\frac{n_N}{n_P} = \lambda\left(\frac{m_N}{m_P}\right)^{\frac{3}{2}} \exp\left(-\frac{\mathcal{J}}{kT}\right) \simeq \lambda e^{-\mathcal{J}/kT}. \tag{15}$$

The first of equations (14) determines λ in terms of n_e. This equation is comparatively easy to solve for both the cases $\lambda \ll 1$ and $\lambda \gg 1$, but is rather awkward when $\lambda \simeq 1$. Throughout the following work the form of solution in which $\lambda \gg 1$ will be used. This is necessary at the highest densities considered (of order 10^{11} gm. per cm.^{3}) but is only a moderate approximation at the lowest densities (of order 10^7 gm. per cm.^{3}) treated in the work of later sections. This procedure is however fully justified by the circumstances that in calculations of the relative abundances of the elements the terms depending on λ are unimportant at the lowest densities (10^7 gm. per cm.^{3}) but are very important at the highest densities (10^{11} gm. per cm.^{3}). Thus our approximation is very good in the range of density where the terms in λ matter, whilst the approximation is only moderate for the range of density in which the terms in λ are unimportant. When $\lambda \gg 1$ and account is taken of the relativistic variation of mass of the electrons in forming the energy levels ϵ_t, the first of equations (14) can be solved to give*

$$\left.\begin{aligned} \ln\lambda &= m_e c^2 \sqrt{(1+x^2)}/kT, \\ x &= (n_e/5{\cdot}87 \,.\, 10^{29})^{\frac{1}{3}}, \end{aligned}\right\} \tag{16}$$

the temperature being here measured in °C. Equations (15) and (16) give

$$\left.\begin{aligned} \ln\left(\frac{n_N}{n_P}\right) &= \frac{m_e c^2}{kT}\left\{\sqrt{(1+x^2)} - \frac{\mathcal{J}}{m_e c^2}\right\} = \frac{m_e c^2 y}{kT}, \\ \text{where} \quad & \\ y &= \sqrt{(1+x^2)} - \mathcal{J}/m_e c^2 = \sqrt{(1+x)^2} - 0{\cdot}51. \end{aligned}\right\} \tag{17}$$

If we again measure T in units of 10^9 °C., the first of equations (17) becomes

$$\log \frac{n_N}{n_P} = \frac{4{\cdot}73}{T} \,.\, 0{\cdot}543y. \tag{18}$$

Substituting in (13) gives

$$\log n_A^Z = 34{\cdot}08 + \frac{3}{2}\log T + \frac{3}{2}\log A + \frac{4{\cdot}73}{T}(Q_A^Z - 0{\cdot}543yZ)$$
$$+ A(\log n_N - 34{\cdot}08 - \frac{3}{2}\log T). \tag{19}$$

The second of equations (16), equations (18) and (19) give the complete scheme of statistical equations in the present problem.

A more useful form of the equation determining the parameter x can be

* S. Chandraskehar, *op. cit.*, p. 393 ; R. H. Fowler, *op. cit.*, p. 652.

obtained by expressing n_e in terms of the density of the material. If we write ρ' for the density excluding the contribution of the free neutrons (in the following discussion the free protons never make an appreciable contribution to the density, but at the highest densities considered below it turns out that the free neutrons do make an appreciable contribution), then to a sufficient approximation we can write

$$n_e \simeq \frac{\rho'}{2m_P} \tag{20}$$

for material not containing hydrogen. This equation follows from the consideration that the assembly must be effectively electrically neutral. This requires that n_e shall be equal to the total positive charge density of the atomic nuclei, which is close to $\rho'/2m_P$ together with a contribution equal to the density of the positrons present in the assembly. The latter term can be calculated from statistical mechanics * and it turns out that its value can be neglected in comparison with $\rho'/2m_P$. Except when ρ' is higher than 10^{10} gm. per cm.^{3}, it is sufficient to put ρ' equal to the total density ρ. The equation for x becomes

$$\rho' = 1{\cdot}98 \times 10^6 x^3. \tag{21}$$

Equation (21) enables x and therefore y to be determined in terms of ρ'. Thus, since Q_A^Z is a known quantity, it follows from (19) that n_A^Z is determined in terms of ρ', T and n_N.

4. *The Helium Zone*

4.1. It would now be possible to return to the physical model discussed in Section 2 and to apply the statistical equations of Section 3. Before doing this, however, it is more convenient to discuss the further properties of these equations and to return to the physical model at a later stage. The two questions that will be considered in the present section and in Section 5 respectively are:

(1) If we regard T, ρ' as Cartesian coordinates, then in the T, ρ'-plane a curve can be drawn with the following important property:—Material at values of T, ρ' represented by any point on one side of the curve is composed almost entirely of helium, whereas material at any point, T, ρ' on the other side of the curve is composed almost entirely of heavy elements, the main mass of the elements in the latter case having atomic weight greater than 50.

(2) Throughout Section 2 it was assumed that nuclear reactions take place sufficiently rapidly when $T > 4 \times 10^9$ °C., $\rho' > 10^7$ gm. per cm.^{3} for the statistical equations to be applicable, whereas when $T < 4 \times 10^9$ °C. it was supposed that the mixture remains frozen. This question will be discussed in detail in Section 5.

It will now be shown when statistical equilibrium occurs that, for a given density ρ', there is a closely determinable value of T in the neighbourhood of which the material changes over from being almost entirely composed of helium to being almost entirely composed of heavy elements. It has to be shown further that in the latter case the most abundant element has atomic weight greater than 50. It is most convenient to assume the latter result to begin with, and then to prove it at the end of the discussion.

Equation (19) can be used to express n_4^2 in terms of y (i.e. ρ'), T, and n_N.

* R. H. Fowler, *op. cit.*, p. 653, equation (1830).

Thus we have

$$\log n_4^2 = 34{\cdot}08 + \frac{3}{2}\log T + \frac{3}{2}\log 4 + \frac{4{\cdot}73}{T}(Q_4^2 - 1{\cdot}086y)$$
$$+4(\log n_N - 34{\cdot}08 - \frac{3}{2}\log T). \quad (22)$$

Equations (19) and (22) enable n_N to be eliminated and give the following relation between n_A^Z and n_4^2:—

$$\frac{\log(An_A^Z) - 34{\cdot}08 - 3/2\log T - 5/2\log A}{A} = \frac{\log 4n_4^2 - 34{\cdot}08 - 3/2\log T - 5/2\log 4}{4}$$
$$+\frac{4{\cdot}73}{T}\left\{\frac{Q_A^Z}{A} - \frac{Q_4^2}{4} + 0{\cdot}543\,y\left(\frac{1}{2} - \frac{Z}{A}\right)\right\}. \quad (23)$$

Consider now the orders of magnitude of the three terms in equation (23).

(i) *Second term on right-hand side of equation* (23). The quantity Q_A^Z/A is the packing fraction of atoms of atomic weight A and charge Z. The values of the various quantities involved are shown in Table II for a number of elements, using experimental values for the packing fraction.*

Table II

	He^4	O^{16}	Si^{28}	Fe^{56}	Cu^{63}	Kr^{82}	Sn^{118}	Pb^{208}
Q_A^Z/A	7·57	8·54	9·01	9·27	9·27	9·22	9·12	8·35
Z/A	0·5	0·5	0·5	0·46	0·46	0·44	0·42	0·40
$Q_A^Z/A - Q_4^2/4$	0·0	0·97	1·44	1·70	1·70	1·65	1·55	0·78
$Q_A^Z/A - Q_4^2/4 + 0{\cdot}543y(1/2 - Z/A)$	0·0	0·97	1·44	1·70 $+0{\cdot}02y$	1·70 $+0{\cdot}02y$	1·65 $+0{\cdot}034y$	...	...

These values taken together with the following values of y for various ρ' show that the second term on the right-hand side of (23) is of order unity when $A > 16$ and $4 < T < 10$:—

Table III

ρ' (gm. per cm.^{3})	10^7	10^8	10^9	10^{10}	10^{11}
x	1·72	3·70	7·96	17·2	37·0
y	1·48	3·32	7·51	16·6	36·5

It will be remembered that the form of the term in y arises from the formulae for relativistically degenerate electrons. The values given in Tables II and III confirm the statement made in Section 3 that the term in y only becomes important when the use of these formulae gives a good approximation for the electron distribution (the condition for the approximation to be a good one is that $x \gg 1$).

(ii) *First term on the right-hand side of* (23).—If all the material were in the form of helium, then

$$\log 4n_4^2 = \log\left(\frac{\rho'}{m_P}\right) = \log\rho' + 23{\cdot}78,$$

whilst if other elements are present

$$\log 4n_4^2 < \log\rho' + 23{\cdot}78.$$

* O. Hahn, S. Flugge and J. Mattauch, *Phys. Z.*, 1941. The value used here for Kr^{82} is less than the value given by these authors. The present value has been chosen to fit smoothly into a maximum of the packing fraction between 60 and 70.

It follows therefore that this term has a magnitude also of order unity when $\rho' < 10^{11}$ gm. per cm.³ and $T > 4$.

(iii) *Left-hand side of* (23).—We now use the assumption that the most abundant heavy element occurs for $A > 50$. Consider for example a most abundant heavy element of atomic weight 60. Table IV gives the values of $\log(60\, n_{60}^{Z})$ for a number of specified values of the left-hand side of (23).

TABLE IV

Left-hand side of (23)	$\log(60\, n_{60}^{Z})$
0·0	$34{\cdot}08 + 3/2 \log T + 5/2 \log 60$
−0·1	$28{\cdot}08 + 3/2 \log T + 5/2 \log 60$
−0·2	$22{\cdot}08 + 3/2 \log T + 5/2 \log 60$

It follows from these numerical values that, when the left-hand side of (23) is zero, $60\, n_{60}^{Z} > \rho'/m_P$ for all $\rho' < 10^{11}$ gm. per cm.³ but that, when this term is $-0{\cdot}2$ then $60\, n_{60}^{Z}$ is extremely small compared with ρ'/m_P for all $\rho' > 10^6$ gm. per cm.³. Accordingly only a very small change in this term is required to change the mixture from being composed almost entirely of heavy elements to being composed almost entirely of helium. This result, taken in conjunction with the fact that only a very small change in T is required to change the second term on the right-hand side of (23) by an amount of order 0·1, shows that the change over from helium to heavy elements must occur at values of T close to that given by putting the right-hand side of (23) equal to zero. Moreover for the purpose of calculating this value of T it is sufficiently accurate to put

$$\log 4 n_4^2 = \log \rho' + 23{\cdot}78.$$

(This equation holds to a good approximation when the mixture is almost all helium, and it can also be used as an order of magnitude relation during the changeover from helium to heavy elements, but it could not be used when T has become sufficiently small for the mixture to be composed almost entirely of heavy elements.) This gives

$$T = \frac{18{\cdot}92\{Q_A^Z/A - Q_4^2/4 + 0{\cdot}543\, y(0{\cdot}5 - Z/A)\}}{10{\cdot}3 + 3/2 \log T + 5/2 \log 4 - \log \rho'}. \qquad (24)$$

When T is greater than the value given by (24), the mixture is composed almost entirely of helium and, when T is less than (24), the mixture is composed almost entirely of heavy elements.

A further question arises as to what values of A and Z should be used in (24). It is clear that the values of A, Z giving the largest $(Q_A^Z - 0{\cdot}543\, yZ)/A$ will give the highest T. This means that the region of heavy elements in the T, ρ'-plane is most extensive when the maximum value of $(Q_A^Z - 0{\cdot}543\, yZ)/A$ with respect to *both* A and Z is used in (24). It follows from Table II that this will occur for $A > 50$, which confirms the statement made above that the most abundant heavy element occurs for $A > 50$. The value of $(Q_A^Z - 0.543\, yZ)/A$ for Cu^{63} gives the following table of values of T, ρ' satisfying (24):—

TABLE V

ρ' (gm. per cm.³)	10^6	10^7	10^8	10^9	10^{10}	10^{11}
T (in units of 10^9 °C.)	4·7	5·5	6·6	8·6	11·2	17·4

These values are sufficiently accurate for the present purpose but some approximation is involved on account of the way $\log T$ is treated in making these calculations. The required values are insensitive to the value of $\log T$ and much arithmetical labour is saved by putting $\log T$ equal to a representative constant (1·0 in the present case). A second order correction is easily inserted after the first order values have been worked out. The pairs of values of T, ρ' given in Table V lie on the curve in Fig. 1 which separates the helium zone from the zone of heavy elements.

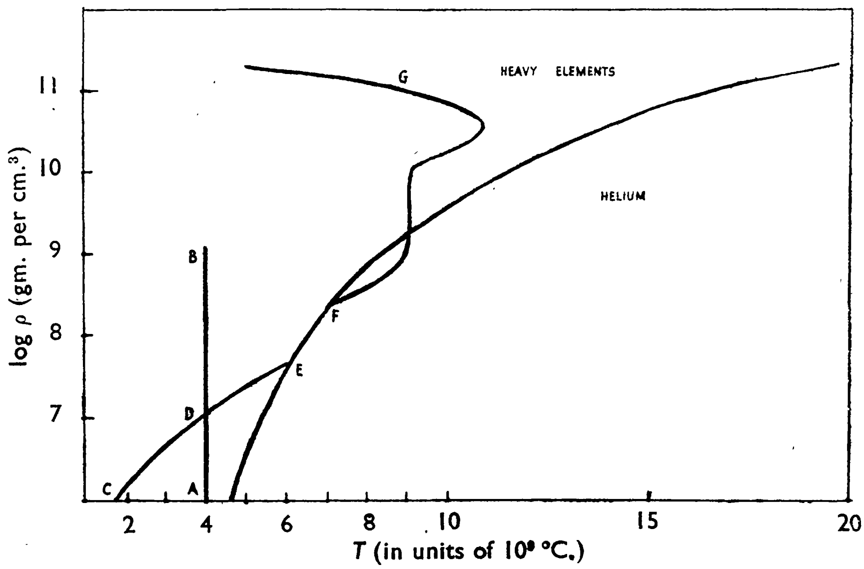

Fig. 1.—*The T, ρ' diagram. The track DEFG is for the case $\alpha = 0{\cdot}24$.*

4.2. *The values of n_N, n_P in the helium zone.*—In the helium zone $\log(4n_4^2) = \log\rho' + 23{\cdot}78$ to good approximation. The value of n_N can be determined in terms of T, ρ' from

$$\log n_N = 34{\cdot}08 + \frac{3}{2}\log T + (\log\rho' - 10{\cdot}3 - \frac{3}{2}\log T - \frac{5}{2}\log 4) - \frac{4{\cdot}73}{T}(7{\cdot}57 - 0{\cdot}271y), \qquad (25)$$

which is derived by putting $Q_4^2/4 = 7{\cdot}57$ and $\log 4n_4^2 = \log\rho' + 23{\cdot}78$ in (22). When n_N has been calculated from (25), the value so obtained can be used in (18) to find n_P. In this way the following numerical values of n_N, n_P were obtained:—

Table VI

$\rho' = 10^7$ gm. per cm.³. $\log n_4^2 = 30{\cdot}18$

T	6	7	8	9	10
$\log n_N$	28·1	29·0	29·7	30·2	30·6
$\log n_P$	27·5	28·4	29·2	29·8	30·2

$\rho' = 10^8$ gm. per cm.³. $\log n_4^2 = 31{\cdot}18$

T	8	9	10
$\log n_N$	30·2	30·7	31·1
$\log n_P$	29·1	29·7	30·2

$\rho' = 10^9$ gm. per cm.³. $\log n_4^2 = 32{\cdot}18$

T	9	10
$\log n_N$	31·5	31·9
$\log n_P$	29·4	30·0

The remarkable feature of Table VI is the large equilibrium values of n_N, n_P that occur for T close to 10. This result is of importance in the discussions of Sections 6 and 9. It will be noticed that at $T=10$ the mass of the neutrons is comparable with, but less than, the mass of the helium. This is the reason why the density of the neutrons was not included in ρ' (the quantity ρ' appears in the formulae only through the relation $n_e \simeq \rho'/2m_P$ so that ρ' must evidently not include the neutrons).

4.3. *The values of n_N, n_P in the heavy element zone.*—Equation (19) can be written in the form

$$\log n_N = 34{\cdot}08 + \frac{3}{2}\log T - \frac{4{\cdot}73}{T}\left(\frac{Q_Z^A}{A} - \frac{0{\cdot}543yZ}{A}\right)$$
$$+ \frac{1}{A}\left\{\log(An_A^Z) - 34{\cdot}08 - \frac{3}{2}\log T - \frac{5}{2}\log A\right\}. \qquad (19')$$

If this equation is applied to the element of maximum abundance then we have

(i) For $10^6 < \rho' < 10^{11}$ the last term on the right-hand side of (19′) may be neglected, since we can put $\log(An_A^Z) = \log\rho' + 23{\cdot}78$.

(ii) The heavy element of maximum abundance occurs when $(Q_A^Z - 0{\cdot}543yZ)/A$ is a maximum with respect to *both* A and Z. The full discussion of this question is left over until Section 7. We may anticipate the result that, when $\rho' \leqslant 10^9$ gm. per cm.^{3}, the required maximum value can be obtained simply by examining the experimentally determined values of Q_A^Z (such as are given in Table II). It then appears that the maximum occurs for A near 60. Thus provided $\rho' \leqslant 10^9$ gm. per cm.^{3} reliable estimates for n_N can be obtained by inserting the experimental values of Q_A^Z and Z/A for Cu^{63} in (19′) (copper serves as an example of an element with atomic weight near 60). The values of n_N so determined can be used in (18) to give n_P. The following values were calculated in this way:—

TABLE VII

$\rho' = 10^7$ gm. per cm.^{3}

T	3	4	5
$\log n_N$	20·8	24·5	26·7
$\log n_P$	19·5	23·5	26·0

$\rho' = 10^8$ gm. per cm.^{3}

T	3	4	5	6
$\log n_N$	21·5	25·0	27·2	28·6
$\log n_P$	18·7	22·9	25·5	27·2

$\rho' = 10^9$ gm. per cm.^{3}

T	3	4	5	6	7	8
$\log n_N$	23·1	26·2	28·1	29·4	30·3	31·1
$\log n_P$	16·7	21·4	24·3	26·2	27·6	28·6

5. *The Conditions for Statistical Equilibrium*

In this section the conditions under which the equations of Section 3 can be applied will be briefly examined. The necessary requirements in the present problem are:

(i) Energy must be statistically distributed among the energy states of translation of each type of particle present (this condition is necessary for the evaluation of the sums in equation (9)).

(ii) Many important nuclear reactions involve the emission of γ-rays (the reactions in which helium is synthesized from hydrogen for example). Statistical equilibrium requires that there shall be detailed balancing between these reactions and the reverse reactions that involve the absorption of a γ-ray. This will only be the case if there is thermodynamic equilibrium between matter and radiation. Furthermore the energies of the γ-rays involved in the nuclear reactions will in some cases be large compared with kT (for the highest temperatures used in the present paper kT is only about 1 M.V.). Thus we require equilibrium to extend up to quanta with energies large compared with kT.

(iii) There must be suitable chain reactions connecting any two pairs of values of A, Z, provided that the nuclei with these pairs of values occur in appreciable abundance. As an example of such a chain reaction, the following known reactions connecting O^{16} and Al^{27} may be noted:

$$O^{16}+He^4 \rightleftarrows F^{19}+H^1, \qquad F^{19}+H^1 \rightleftarrows Ne^{20}+h\nu,$$
$$Ne^{20}+He^4 \rightleftarrows Na^{23}+H^1, \qquad Na^{23}+H^1 \rightleftarrows Mg^{24}+h\nu,$$
$$Mg^{24}+He^4 \rightleftarrows Al^{27}+H^1.$$

Unfortunately there is not sufficient laboratory evidence available to give a known chain reaction for every pair of values of A, Z. Two difficulties arise in attempting to apply only known reactions to the present problem. First, there is far too little published evidence concerning nuclei with atomic weights greater than 50, and, second, at the high densities involved in the present problem it is to be expected that triple collisions between nuclear particles will be important, whereas in the laboratory densities are far too low for these processes to be noticeable. In these circumstances it has been decided to make the following assumptions concerning the chain reactions :—

(a) That a chain reaction is available between any two pairs of values of A, Z and that only neutron, proton, and α-particles reactions are required to establish the chain (these reactions include $n-\alpha$, $n-p$, $n-\gamma$, $n-2n$, $p-\alpha$, $p-\gamma$, $p-n$, $\alpha-n$, reactions together with their inverses).

(b) That for 5 M.V. protons* and α-particles the cross-sections of all reactions occurring in these chains exceed 10^{-28}.

(c) That statistical equilibrium cannot be established without the proton reactions, which are required not only to enable the chain reactions to take place but also to satisfy condition (iv) given below.

(iv) The equilibrium between protons and neutrons requires a sufficiently rapid β-process to occur. The importance of a particular β-reaction depends not only on the lifetime for the process but also on the abundance of the nucleus involved.

The conditions (i), (ii), (iii) will now be considered in further detail, while the discussion of condition (iv) will be postponed until the results of Sections 6, 7 and 8 are available. The first question to decide is the length of time that can be allowed for statistical equilibrium to be established. The work of Section 9 may be anticipated which shows that condition (iv) taken in conjunction with the properties of collapsing stars fixes a time-scale of about 100 sec. for the most rapid stages in the collapse of a star (it will be seen that the early stages of collapse are much slower than this).

Condition (i).—This requirement will be satisfied provided all types of particle experience an appreciable number of collisions per second. It is easy to show

* 1 M.V.≡million electron volts $=1\cdot59\times10^{-6}$ergs.

that this must be the case. Taking firstly the charged particles, the collision cross-section is of order 10^{-16} and the number of collisions experienced by a particle is of order $10^{-16}\, vn$ per sec., where v, n are suitably averaged values of the velocity and particle density respectively. For the values of T, ρ' under consideration, $v \simeq 10^9$ cm. per sec., $n > 10^{29}$, so that the number is of order 10^{22} per sec. Thus condition (i) is satisfied by a very large margin for these particles.

The collision cross-section for neutrons is much smaller than for charged particles. A rough estimate of 10^{-24} for the elastic collision cross-section of the neutrons still gives a collision rate, however, of 10^{14} per sec. Thus even in this case the number of collisions is amply large enough for statistical equilibrium to be established among the states of kinetic energy of the neutrons.

Condition (ii).—To establish this condition it has to be shown that a sufficiently large interchange of energy takes place between matter and radiation. Moreover this interchange must take place up to energies large compared with kT. The phrase "sufficiently large" may be taken to have the following precise meaning:— that the amount of thermal energy of the material converted into radiant energy per sec. per cm.³ in any frequency range ν to $\nu + d\nu$ must be large compared with the equilibrium energy density of radiation given by the Planck formula

$$\frac{8\pi h\nu^3 d\nu}{c^3\{\exp(h\nu/kT)-1\}}. \tag{26}$$

Under these conditions detailed balancing will occur and equal quantities of radiant energy will be converted into thermal energy. It is to be expected that it will be most difficult to establish this requirement when $h\nu \gg kT$, and this case will now be considered.

The number of electrons with energy greater than E_0 per cm.³ is given by

$$\frac{2n_e}{\pi^{\frac{1}{2}}(kT)^{\frac{3}{2}}}\int_{E_0}^{\infty} \exp(-E/kT) E^{\frac{1}{2}}\, dE,$$

where n_e is the number of electrons per cm.³. (This statistical equilibrium formula can be used since condition (i) is satisfied.) For the case $E_0 \gg kT$, this expression can be written to sufficient accuracy as

$$2n_e\left(\frac{E_0}{\pi kT}\right)^{\frac{1}{2}} \exp\left(-\frac{E_0}{kT}\right). \tag{27}$$

Now there is an appreciable probability that an electron will lose almost the whole of its kinetic energy in the Bremstrahlung emission of a single quantum.* For collisions with a nucleus of charge Z the cross-section for this process is about $5 \times 10^{-27} Z^2$. Combining this with (27), it is easy to see that the number of quanta with energy greater than E_0 emitted by the electrons per cm.³ per sec. is of order

$$10^{-26} n_e v_e \left(\sum_{A,Z} Z^2 n_A^Z\right)\left(\frac{E_0}{kT}\right)^{\frac{1}{2}} \exp\left(-\frac{E_0}{kT}\right), \tag{28}$$

where v_e is the velocity of electrons with energy E_0, and the summation is taken over all nuclei present in the material. Now the number of quanta with energy greater than E_0 per cm.³ under thermodynamic equilibrium is

$$\frac{8\pi}{c^3}\int_{\nu_0=E_0/h}^{\infty} \frac{\nu^2 d\nu}{\exp(h\nu/kT)-1} \simeq \frac{8\pi E_0^2 kT}{c^3 h^3} \exp\left(-\frac{E_0}{kT}\right), \tag{29}$$

* W. Heitler, *Quantum Theory of Radiation*, p. 170, Oxford, 1936.

when $E_0 \gg kT$. An order of magnitude estimate for the time required for equilibrium to be set up can now be obtained by dividing (29) by (28). If we put $A=4$, $Z=2$, $2n_e=n_4^2=10^{30}$ per cm.^{3}; $v_e=10^{10}$ cm. per sec., $E_0=20$ M.V., $kT=0{\cdot}3$ M.V., this gives a time of order 10^{-12} sec. Thus condition (ii) is also satisfied by a very large margin.

Condition (iii).—Proton, α-particle and neutron reactions will now be considered subject to the assumptions (*a*), (*b*), and (*c*) made above. Take proton reactions first. Since condition (i) is satisfied the number of protons per cm.^{3} with energy lying between E and $E+dE$ is

$$\frac{2n_P E^{\frac{1}{2}} dE}{\pi^{\frac{1}{2}}(kT)^{\frac{3}{2}}} \cdot \exp\left(-\frac{E}{kT}\right).$$

Thus the probability of a given nucleus of atomic weight A and charge Z experiencing a proton reaction in a time δt is

$$\frac{2\delta t n_P}{\pi^{\frac{1}{2}}(kT)^{\frac{3}{2}}} \int_0^\infty \sigma_A^Z(E) v_P(E) \exp(-E/kT) E^{\frac{1}{2}} dE, \tag{30}$$

where $\sigma_A^Z(E)$ is the proton reaction cross-section of the given nucleus, and $v_P(E)$ is the velocity of protons of energy E. The reaction cross-section increases rapidly with E (for the range of E that makes an appreciable contribution to the integral (30)) on account of the Coulomb barrier of the nucleus.

Experimental cross-sections are nearly always measured at one or two discrete values of E, and it is not possible to use measured values over the whole range of E in (30). This integral can be estimated so far as order of magnitude is concerned, however, by taking $E^{\frac{1}{2}} . v_P(E) . \sigma_A^Z(E)$ outside the integral at a particular value E_0 of E, where E_0 is chosen so that $\sigma_A^Z(E)$ is known from experiment. (This remark assumes that the reaction under consideration is one for which experimental measurements are available. As pointed out in the following paragraph, only a small proportion of the reactions of interest in the present problem fall in this class.) To avoid overestimating the integral a lower limit of E_0 is then used instead of zero. That is, (30) can be estimated as

$$2\delta t n_P . v_P(E_0)\sigma_A^Z(E_0) . \left(\frac{E_0}{\pi kT}\right)^{\frac{1}{2}} \exp\left(-\frac{E_0}{kT}\right). \tag{31}$$

In the majority of reactions of importance in the present problem no experimentally determined cross-sections are available and it is necessary to fall back on the assumption (*b*) made above. According to this assumption when $E_0=5$ M.V. the cross-section $\sigma_A^Z(E) \geqslant 10^{-28}$ for all A, Z. An estimate for the probability of proton reactions can then be obtained by inserting these values in (31). (If we put $\sigma_A^Z(E_0)=10^{-28}$ the probability will be underestimated. It is desirable to underestimate the probabilities rather than to overestimate them.) It follows immediately from (31) that a nucleus A, Z will have a proton reaction in a time of less than 100 seconds provided n_P is greater than the values given in the following table:—

TABLE VIII

T	4	5	6	7	8
$\log n_P$	22·2	21·0	20·2	19·5	19·2
$\log n_4^2$	22·5	21·3	20·5	19·8	19·5

The assumption (*b*) also gives a cross-section $\geqslant 10^{-28}$ when $E_0=5$ M.V. for α-particle reactions, so that by allowing for the difference between the thermal velocities of the protons and α-particles we obtain immediately the corresponding criterion for n_4^2, in order that a nucleus A, Z shall experience an α-particle reaction in a time of less than 100 seconds.

The cross-sections for neutron reactions are of the order of, or less than, 10^{-24}. Moreover the neutrons do not have to penetrate a Coulomb barrier, so that the incident neutrons may have kinetic energies of order kT. Thus the probability of neutron reactions is far higher than the probability of proton reactions (this follows since n_N is always greater than n_P).

It was assumed throughout Section 2 that statistical equilibrium applies when $T > 4 \times 10^9$ °C. but not when $T < 4 \times 10^9$ °C. This statement follows immediately by comparing the values of $\log n_P$ in Table VII with the requirements for statistical equilibrium given in Table VIII. It is seen that the value of n_P becomes too small as T decreases below 4×10^9 °C. for condition (iii) to hold. This result is insensitive to the value of ρ' and may be applied over the range $10^6 < \rho' < 10^8$ gm. per cm.^{3}.

Condition (iv).—This condition is discussed below in Section 8.

6. *The Track of the Stellar Material in the Helium Zone*

6.1. In Sections 3, 4 and 5 the equations of statistical equilibrium have been set up and certain of their general properties discussed. It is now proposed to return to the model of a collapsing star and to apply the results of Sections 3, 4 and 5 during the collapsing phase. It will be assumed throughout the present and following sections that a state of rotational instability has not been reached. In this connection it will be recalled that the onset of rotational instability can be arranged to occur at any stage we please by suitably adjusting the initial rotational velocity (see Table I).

At any particular time the values of T, ρ' for the material at the centre of a collapsing star can be represented as a point in the T, ρ'-plane. This point moves with time and the work of the present and following sections enables its track to be found.

The equations of statistical equilibrium can be applied when $T > 4 \times 10^9$ °C. (During the early stages of contraction the time available for equilibrium to be set up is much greater than the value of 100 seconds mentioned in Section 5. Thus statistical conditions may occur before T rises to 4×10^9 °C. *during the contraction*. On the other hand during expansion after rotational instability has taken place a time of several seconds only is available for establishing equilibrium. Thus on expansion the mixture will remain frozen when T falls below 4×10^9 °C. This question is discussed in detail in Section 8.) For the star of mass 10 $M_\odot$ considered in Table I the value of ρ' is about 5×10^6 gm. per cm.^{3} when $T = 4 \times 10^9$ °C., while for stars of other masses the corresponding values of ρ' are given approximately by $5 \times 10^7 (M_\odot / M)$ gm. per cm.^{3} (the central density of a "normal" star is approximately proportional to $1/M$). Thus the values of ρ', for the mass $M < 50\, M_\odot$ but exceeding Chandrasekhar's limit, lie in the range $10^6 < \rho' < 5 \times 10^7$ gm. per cm.^{3} at the time T reaches 4×10^9 °C. It follows therefore that the material at the centre of a collapsing star crosses the line $T = 4$ (T now in units of 10^9 °C.) at a point lying between A and B in Fig. 1. For the sake of definiteness a star will be considered in the following work that crosses this line at $\rho' = 10^7$ gm. per cm.^{3} (marked D in Fig. 1). This requires a star of mass close to 5 $M_\odot$. To the left of the line $T=4$ it follows from (2) (which applies in this region of the T, ρ'-diagram) that the track of the material at the centre of the star is given by

$$\rho' = 10^7(T/4)^3 \text{ gm. per cm.}^3. \qquad (32)$$

This portion of the track of the material is shown in Fig. 1 by the curve CD.

From Fig. 1 it is seen that on passing the point D (moving from left to right in the figure) the stellar material enters the heavy element zone. Before reaching D the material is composed largely of helium, but on passing D the helium is converted almost entirely into heavy elements (it will be recalled that the helium arises by synthesis from hydrogen; this process occurring while the star is in a "normal" equilibrium state). Now the synthesis of heavy elements from helium gives an important energy yield, since the packing fractions for the heavy elements are larger than the packing fraction of helium. Indeed it is easy to see from the values of the packing fraction given in Table II that the total amount of energy released in the conversion of the helium must be of order $Mc^2/500$. This value may be compared with the total energy generated by the synthesis of helium from hydrogen during the "normal" equilibrium state. This latter energy is of order $Mc^2/125$, so that the conversion of helium to heavy elements near the point D in Fig. 1 yields about 1/4 of the total energy generated while the star was in a "normal" state. This amount is large enough for a new "collapsed" equilibrium state to be set up, in which the material at the centre of the star is in the neighbourhood of the point D in the T, ρ'-diagram. The lifetime of the star in this new equilibrium state can only be a small fraction, however, of the lifetime in the normal state. This is due partly to the smaller total energy yield and partly to the increased rate of radiation from the surface (this is shown by (5) which is proportional to $R^{-1/2}$ so that the rate of radiation is considerably larger in the collapsed equilibrium state than in the normal state). In addition there is an important third reason explained in the following paragraph that reduces the lifetime of the collapsed equilibrium state by an appreciable factor.

When the lifetime of the collapsed equilibrium state is over, the material at the centre of the star will pass the point D in the T, ρ'-diagram, and further collapse of the star (further collapse must occur when energy generation ceases) will lead to the material evolving further along the track given by (32). Now at the point E in Fig. 1 the material reaches the helium zone, which means that there must be a *reconversion* of the material from heavy elements back into helium. In order to effect this reconversion energy must be supplied to the material in an amount equal to the energy that was obtained on passing the point D. There are two processes that contribute to this energy:

(1) The collapse of the star yields gravitational energy.

(2) Since the central regions of a star have a temperature higher than material at greater distances from the centre it follows that the first material in the star to pass the point D in the T, ρ'-diagram will be material lying in the central regions. This remark taken with the fact that once the point D has been passed the evolution from D to E will be comparatively rapid, shows that material at the centre will have reached E before an appreciable fraction of the remaining material of the star has passed D. Thus owing to the star not being at a uniform temperature the process of conversion of helium to heavy elements and the reverse process

of reconversion back to helium will be mixed together, and a proportion of the energy released in the first process will be available for use in the reverse process.

Although it is certain that gravitational energy is required for the reconversion of the heavy elements back to helium it is not possible to estimate the amount quantitatively without calculating the amount of energy that is available through the process (2) of the previous paragraph. This is difficult and will not be attempted. For the present purpose it is sufficient to know that gravitational energy is required before the stellar material can enter the helium zone. Thus when the track of the material reaches E in the T, ρ'-diagram, gravitational energy must be supplied before the track can enter the helium zone. Now the value of ρ' increases whilst the gravitational energy is being supplied, so that the point in the T, ρ'-diagram (representing the material) must move during the time that this energy is made available. It follows therefore that during the supply of the gravitational energy the track of the material *must move along the curve that divides the heavy element zone from the helium zone.* This portion of the track is marked EF in Fig. 1. The important result follows immediately that since the curve EF does not satisfy (32) the hydrostatic equation of support (1) cannot now be satisfied even as an approximation. Stated in more physical terms, when the point E is reached the temperature cannot increase at a sufficiently rapid rate for (1) to be approximately satisfied because the gravitational energy supplied by contraction is swallowed up by the reconversion of heavy elements to helium. We may anticipate the results of Section 9 by stating that the evolution of the material along EF only takes a time of several seconds. Thus the collapse of the star becomes extremely swift when the point E is reached. As will be shown in 6.2 and Section 7 this rapid collapse is also maintained beyond F. Thus the point E is of the nature of a critical point; before E is reached the collapse of the star is very slow (occupying a time of several thousand years at least) whilst beyond E the collapse is catastrophic, the whole evolution then being carried out in a time of a few seconds.

6.2. The next step in the investigation is to trace the track of the material through the helium zone. In this work we shall be concerned with densities that rise appreciably above 10^9 gm. per cm. These high densities necessitate an important change in the formula for the internal energy of the material. Up to now it has been sufficiently accurate to take the thermal energy of the material as $3\mathfrak{R}\rho T/2\mu$ per cm.^{3}. This expression together with the energy density aT^4 of the radiation gave the total energy per cm.^{3} used in the above work. At sufficiently high densities, however, the quantity $3\mathfrak{R}\rho T/2\mu + aT^4$ becomes appreciably less than the energy of degeneracy of the electrons. The latter energy is given by the expression *

$$2\pi m_e^4 c^5 x^4/h^3, \tag{33}$$

when $x \gg 1$ (powers of x smaller than x^4 being neglected). When (21), which expresses x in terms of ρ', is used (33) takes the form

$$1{\cdot}53 \times 10^{15}(\rho')^{\frac{4}{3}}. \tag{33'}$$

It is seen that (33′) exceeds $3\mathfrak{R}\rho' T/2\mu$ when $\rho'/\mu = 10^9$ gm. per cm.^{3} $T \leqslant 10^{10}$ °C. The contribution (33′) to the energy of the material arises from the Pauli exclusion principle which requires the average energy of the degenerate electrons to be large

* S. Chandrasekhar, *op. cit.*, p. 392.

when the density is high (this is quite apart from the value of the temperature T, and would apply even if T were zero). In the following discussion the ordinary thermal energy of the material and the energy of radiation will both be neglected in comparison with (33′).

The results given in Table VI show that T can only rise to 10^{10} °C. if n_N takes a value comparable with n_4^2. Thus in order for T to rise to 10 in the T, ρ'-diagram an appreciable quantity of helium must be converted into free neutrons. Now a large quantity of energy has to be supplied to the stellar material to make this conversion possible. This energy can only be derived from the gravitational energy supplied by the collapse of the star. Thus the situation is similar to that occurring between the points E and F. That is, the energy supplied by contraction is largely absorbed in nuclear reactions and only a small proportion goes into raising the temperature of the material. Between E and F the energy is absorbed by the reconversion of heavy elements to helium, whereas inside the helium zone the energy is used up in producing free neutrons. This is the reason why the rapid collapse of the star between F and E must continue after the material has entered the helium zone.

An order of magnitude estimate for the amount of energy released in the collapse of the star can be obtained in terms of ρ' and M. The "energy released" in the present sense is meant to include energy that is available for absorption in any of the following three ways *:—

(1) In increasing the value of T. This energy not only includes the increased thermal energy of the material and the increased energy of radiation but also the loss of energy by neutrino emission in the β-processes. The loss of energy through neutrino emission would have the effect of reducing T if energy were not absorbed to make good the loss.

(2) In increasing the energy of rotation of the star.

(3) In increasing the ratio n_N/n_4^2.

As the effect of neutrino emission is more conveniently dealt with at a later stage (Section 8) this sink of energy will be neglected for the moment. In addition the absorption of energy in increasing the energy of rotation only becomes important when the star is nearing the state of rotational instability. Thus, since in the present discussion the star is not regarded as being in any imminent danger of becoming rotationally unstable, the energy absorbed in (2) may also be neglected.

The gravitational potential energy $-\Omega$ of the star is given by

$$-16\pi^2 G \int_0^R \rho(r) r \, dr \int_0^r \rho(r')(r')^2 \, dr', \tag{34}$$

where R is the radius of the star. If we assume that whatever the values of M, R the star is always a member of a single homologous series, it follows from (34) that Ω is of the form

$$\Omega(M, R) = BM^2/R, \tag{35}$$

where B is independent of M, R. This assumption would not be permissible in an accurate calculation but can properly be used in making an order of magnitude estimate for Ω. At first sight it might seem that the energy released (in the sense

* Once the point E has been passed the amount of energy radiated from the surface of the star is negligible compared with the energy absorbed in these processes.

of the previous paragraph) in a contraction of the star from R to $R-\delta R$ should be given by $\Omega(M, R-\delta R)-\Omega(M, R)$ but this is not the case because energy has to be supplied merely in order to increase the density of the stellar material. Thus according to (33′) the total "energy of degeneracy" of the star is given by

$$6\cdot 1 \times 10^{15}\pi \int_0^R (\rho')^{\frac{4}{3}} r^2\, dr.$$

In an order of magnitude calculation the contribution of the neutrons to ρ can be neglected, and we may put $\rho'=\rho$ in (36). It then follows by using the assumption of the homologous series that the energy of degeneracy is of the form

$$CM^{\frac{4}{3}}/R, \tag{37}$$

where C is independent of M, R.

Since (35) and (37) depend differently on M there must be some value of the mass for which these two quantities are equal. According to Chandrasekhar * this occurs when M is equal to the limiting mass $5\cdot 75\ M_{\odot}/\mu_e^2 = M_{cr}$ say. Thus for $M > M_{cr}$ we have

$$\frac{\Omega}{\text{Total energy of degeneracy}} \simeq \left(\frac{M}{M_{cr}}\right)^{\frac{2}{3}}.$$

Moreover, since the total energy released is given by

$$\Omega - [\text{Total energy of degeneracy}],$$

it follows that

$$[\text{Total energy released}] = [\text{Total energy of degeneracy}].\ [(M/M_{cr})^{\frac{2}{3}} - 1]. \tag{38}$$

Now the energy of degeneracy *per cm.*³ is given by (33′) so that the energy released *per cm.*³ can be estimated as

$$1\cdot 53 \times 10^{15}(\rho')^{\frac{4}{3}}\{(M/M_{cr}{}^{\frac{2}{3}}) - 1\}.$$

This expression can be replaced by the two equations

$$\left.\begin{aligned} [\text{Energy released per cm.}^3] &= 1\cdot 53 \times 10^{15}\alpha(\rho')^{\frac{4}{3}}, \\ \alpha &\simeq (M/M_{cr})^{\frac{2}{3}} - 1. \end{aligned}\right\} \tag{39}$$

The purpose of this apparently trivial step is that values of α can be specified and the first of equations (39) can be regarded as an *accurate* equation for the energy released. The second of equations (39) then becomes an equation that determines M *approximately* in terms of the specified α. In other words, instead of specifying M and calculating the energy release approximately we can specify the energy release and calculate the mass approximately. Since it is the energy release that is required in the present work and not the mass the second alternative is preferable. It might seem that the procedure of specifying the energy release renders the above investigation superfluous. This is not the case, however, for we have to be sure that the values we specify are reasonable values and this cannot be decided unless the above discussion of order of magnitude is given.

The energy absorbed in increasing the thermal energy of the stellar material and the energy density of radiation is small compared with (33′). Thus since the energy absorbed in increasing the rotation of the star and in making good the loss of energy by neutrino emission is being neglected in the present discussion,

* S. Chandrasekhar, *op. cit.* p. 424.

effectively the whole of the energy released is to be regarded as absorbed in the production of free neutrons. The energy required to produce one free neutron is about $m_N c^2/125$. Thus the free neutron density is given by dividing the energy released per cm.[3] by $m_N c^2/125$. This gives

$$\log n_N = 20{\cdot}1 + \frac{4}{3}\log\rho' + \log\alpha. \tag{40}$$

This equation plays a very important part in the following discussion. The value of M is given approximately in terms of α by the second of equations (39). A selection of corresponding values of M and α are given in the following table:—

TABLE IX

M/M_{cr}	1	2	3	5	7	10	15
α	1	0·59	1·08	1·92	2·66	3·64	5·08

If the value of $\log n_N$ given by (40) is substituted in (25) the following equation between T and ρ' is obtained:—

$$T = \frac{4{\cdot}73(7{\cdot}57 - 0{\cdot}271y)}{11{\cdot}03 - 13/12\log\rho' + 9/8\log T - \log\alpha}. \tag{41}$$

Equation (41) gives the track of the stellar material in the helium zone. We now see that the neglect of the rotational energy and the energy lost in neutrino emission has very little effect on the track of the material. Absorption of energy by these processes could be included by suitably altering the value of α in (41), but since α appears only through its logarithm the consequent change in T would be very small.

It is convenient in numerical calculations based on (41) to put $9/8\log T$ equal to a representative constant. For the case $\alpha = 0{\cdot}24$, $M/M_{cr} = 1{\cdot}4$, put $9/8\log T = 1$. The values of T corresponding to a set of values of ρ' are then easily obtained.

TABLE X

$\alpha = 0{\cdot}24$, $M/M_{cr} \simeq 1{\cdot}4$.

ρ' (gm. per cm.^{3})	10^8	10^9	10^{10}
y	3·32	7·5	16·6
T (in units of 10^9 °C.)	7·9	9·0	8·0

The error introduced by putting $\log T = 8/9$ is evidently very slight. Similar sets of values can be obtained for other values of α (that is, for other values of M/M_{cr}). An important result emerges from the values given in Table X. The two points $\rho' = 10^8$, $T = 7{\cdot}9$; $\rho' = 10^9$, $T = 9{\cdot}0$ lie within the helium zone (see 4.1), but the point $\rho' = 10^{10}$, $T = 8{\cdot}0$ lies in the heavy element zone. This means that the track of the stellar material passes through the helium zone *and moves out again into the heavy element zone.* This is a general result and does not depend on the particular value of α used in Table X. Thus if we are to follow the track of the material at densities higher than 10^9 gm. per cm.^{3} it is necessary also to work out the track in the heavy element zone. This question will now be considered.

7. *The Track of Stellar Material in the Heavy Element Zone*

If A, Z are chosen so that $(Q_A^Z - 0{\cdot}543yZ)/A$ is a maximum with respect to *both* A and Z then in the heavy element zone

$$\rho' \simeq A n_A^Z m_P,$$

because, as shown in Section 4, these values of A, Z give the most abundant heavy element. If this maximum value is denoted by $(Q_A^Z/A - 0{\cdot}543yZ/A)_{\max}$ then we obtain from equation (19′)

$$\log n_N = 34{\cdot}08 + \frac{3}{2}\log T - \frac{4{\cdot}73}{T}\left(\frac{Q_A^Z}{A} - \frac{0{\cdot}543yZ}{A}\right)_{\max}, \tag{42}$$

since the last term on the right-hand side of (19′) is negligible when $\rho' \simeq A n_A^Z m_P$ (it will be recalled that for the most abundant element $A > 50$). When $\rho' < 10^9$ gm. per cm.[3] the contribution of the y term is small and

$$\left(\frac{Q^Z}{A} - \frac{0{\cdot}543yZ}{A}\right)_{\max} \simeq \left(\frac{Q_A^Z}{A}\right)_{\max}.$$

In this case $(Q_A^Z/A)_{\max}$ is determined, as in Section 4, by using the values of Q_A^Z measured in the laboratory. At very high densities, however, the term in y becomes important and it is no longer possible to obtain $(Q_A^Z/A - 0{\cdot}543yZ/A)_{\max}$ directly from laboratory data. It is then necessary to use an expression for Q_A^Z/A as an explicit function of A, Z. Such an expression due to Weizsäcker is available and can be written in the form *

$$\frac{Q_A^Z}{A} = 14{\cdot}9 - 21\frac{(A-2Z)^2}{A^2} - \frac{14{\cdot}2}{A^{\frac{1}{3}}} - \frac{0{\cdot}625Z^2}{A^{\frac{4}{3}}} \tag{43}$$

The first term on the right-hand side is the main term; the second term is a correction arising from the Fermi-Dirac statistics satisfied by the proton and neutron; the third term is a surface effect; the fourth term represents the electrical energy of the nucleus arising from the Coulomb forces between the protons. This expression is in good agreement with the measured values of Q_A^Z/A over the whole range of the periodic table. Using (43) we have

$$\frac{1}{A}(Q_A^Z - 0{\cdot}543yZ) = -(5{\cdot}1 + 14{\cdot}2A^{-\frac{1}{3}}) + \frac{Z}{A}(84 - 0{\cdot}543y) - \frac{Z^2}{A^2}(84 + 0{\cdot}625A^{\frac{2}{3}}). \tag{44}$$

The maximum value of (44) with respect to Z is given by

$$\frac{2Z}{A} = \frac{1 - 0{\cdot}00647y}{1 + 0{\cdot}00744A^{\frac{2}{3}}} \tag{45}$$

and is equal to

$$-(5{\cdot}1 + 14{\cdot}2A^{-\frac{1}{3}}) + \frac{21(1 - 0{\cdot}00647y)^2}{1 + 0{\cdot}00744A^{\frac{2}{3}}}. \tag{46}$$

The maximum value of (46) is given by putting $A = A_{\max}$, where

$$A_{\max} = 45{\cdot}44\,\frac{(1 + 0{\cdot}00744A^{\frac{2}{3}})^2}{(1 - 0{\cdot}00647y)^2}. \tag{47}$$

Substituting $A = A_{\max}$ in (45) and (46) gives

$$\left.\begin{aligned} \left(\frac{Q_A^Z}{A} - \frac{0{\cdot}543yZ}{A}\right)_{\max} &= \frac{954{\cdot}2}{A_{\max}}(1 + 0{\cdot}00744A_{\max}^{\frac{2}{3}}) - (5{\cdot}1 + 14{\cdot}2A_{\max}^{-\frac{1}{3}}) \\ \left(\frac{2Z}{A}\right)_{\max} &= \frac{1 - 0{\cdot}00647y}{1 + 0{\cdot}00744A_{\max}^{\frac{2}{3}}}, \end{aligned}\right\} \tag{48}$$

* H. A. Bethe and R. F. Bacher, *Rev. mod. phys.*, **8**, 82, 165, 1936.

where $(2Z/A)_{max}$ is the value of $(2Z/A)$ for the element of maximum abundance.

It follows from the equations derived above that A_{max} and $(Q_A^Z/A-0{\cdot}543yZ/A)_{max}$ are functions of ρ'. The following table gives a selection of values of A_{max}, $(2Z/A)_{max}$, and $(Q_A^Z/A-0{\cdot}543yZ/A)_{max}$ for various ρ'. (It may be noted that the free neutrons cease to obey "classical" statistics when the density exceeds the values given in this table. The statistical equations would require modification before they could be applied to degenerate neutrons.)

TABLE XI

$\log \rho'$	10·08	11·04	11·27	11·49	11·59	11·79
y	18·4	38·2	45·7	54·1	58·6	68·0
A_{max}	75	110	130	160	180	240
$\left(\frac{Q_A^Z}{A}-\frac{0{\cdot}543yZ}{A}\right)_{max}$	4·67	2·10	0·84	−0·45	−1·07	−2·27
$\left(\frac{2Z}{A}\right)_{max}$	0·78	0·65	0·59	0·53	0·50	0·44

The above values show that for sufficiently high ρ' the most abundant element in the heavy element zone lies at the top end of the periodic table. This is one of the most important results derived in the present paper. It arises directly from the term in y which is due to the degeneracy of the electrons. The values given for $(2Z/A)_{max}$ show that at these high densities the ratio of neutrons to protons present in the heavy elements is far higher than is found in the laboratory.

The discussion of the track of the stellar material in the heavy element zone of the T, ρ'-diagram is similar to that given in 6.2 for the helium zone, except that in place of equation (25) for the helium zone we now have equations (42), (47) and (48). These latter equations together with (40) determine the track of the material in the heavy element zone. The equation for T corresponding to (41) is

$$T=4{\cdot}73\frac{\{(954{\cdot}2/A_{max}).(1+0{\cdot}00744A_{max}^{\frac{2}{3}})-(5{\cdot}1+14{\cdot}2A_{max}^{-\frac{1}{3}})\}}{13{\cdot}98+3/2\log T-4/3\log\rho'-\log\alpha}, \qquad (49)$$

where A_{max} is given in terms of y (and consequently in terms of ρ') by (47). Thus (49) gives T as a function of ρ' and therefore gives the required track of the stellar material in the T, ρ'-diagram. It is again convenient to put $\log T$ equal to a representative constant in the numerical work. The following table gives values of T, n_N, n_P for a set of values of ρ' for the case $\alpha=0{\cdot}24$:—

TABLE XII

$\alpha=0{\cdot}24$, $\log T=0{\cdot}85$.

$\log \rho'$	10·08	10·56	11·04	11·27
T	9·09	10·78	8·64	4·73
$\log n_N$	32·9	33·6	34·2	34·5
$\log n_P$	27·7	27·3	22·8	9·7

The outstanding feature of this table is the very rapid decrease in n_P when ρ' rises to 10^{11} gm. per cm.^{3}. The reason why n_P decreases so rapidly is that equations (40) and (42) prevent T rising appreciably above 10 (if T were to rise appreciably above 10 then equation (42) would give a value of n_N that was too large to be consistent with (40). This limitation in temperature is a direct result of the absorption in free neutron production of the energy released in the collapse of the star). Thus as ρ' increases the second term on the right-hand side of (18) increases until it becomes comparable with $\log n_N$. At this stage both n_P and T decrease very rapidly. Thus beyond this stage it is probably unsafe to employ the equations of statistical equilibrium. This means that the relative abundances of the elements can no longer be calculated on a statistical basis. The problem evidently becomes of extreme complexity, for nuclear reactions can continue to be important for some elements and can become unimportant for others. Indeed it seems possible that some elements may become frozen simply as a result of a sufficiently large increase in density.

The values given in Tables X and XII depend on the value $\alpha = 0{\cdot}24$ used in the computations. For other values of α a curve similar to that shown in Fig. 1 can be obtained. In particular the values of T, ρ' at the point where statistical equilibrium breaks down depend on α. The values of T, ρ' near this break-down point are given in the following table for a selection of values of α:—

Table XIII

α	1·07	2·69	3·89	5·49
T (at break-down point)	9·79	11·58	12·50	14·55
ρ' (at break-down point)	11·27	11·49	11·59	11·79
Atomic weight of most abundant element at break-down point	130	160	180	240

The values given in Table XIII show the very important result that for sufficiently large α the most abundant element at the break-down point lies at the upper end of the periodic table.

In following the track of the material in the T, ρ'-diagram we have referred explicitly to the material at the centre of the star. The above discussion enables the case of material not at the centre to be immediately included, for the only difference that can arise is through a difference in the value of the parameter α. It is to be expected that α will have a different value in different parts of the star (α could only be constant throughout the star if there was a very special distribution of material in the star. There seems to be no reason why a rapidly collapsing star should in general satisfy such a special distribution). This means that there will be a variation in both the statistical break-down point and the element of maximum abundance at the break-down point as we pass from one part of the star to another.

The discussion of the track of the stellar material given in the present and previous sections depends on equation (40). This equation applies only so long as insufficient energy is released by the contraction of the star to convert effectively the whole of the material into neutrons. If, however, the collapse of the star continues after material reaches the break-down point, a stage must eventually be reached at which equation (40) no longer holds. At this stage energy is again available to raise the temperature of the material and this has the effect of increasing n_P. Thus it is possible for statistical equilibrium to be resumed if the contraction of the star should proceed sufficiently far before rotational instability occurs. It is unnecessary for the purpose of the present paper to consider the properties of the material in this new state, and this will not be attempted.

8. *The Chemical Composition of Material thrown off by certain Rotationally Unstable Collapsing Stars*

8.1. *Preliminary Remarks.*—In the present section the results so far obtained will be used to give a discussion of the relative abundances of the elements distributed in interstellar space by the process of rotational instability. Since we are considering a star with mass appreciably greater than Chandrasekhar's limit it follows that collapse must continue until rotational instability occurs.* For the sake of definiteness a star of initial mass 20 $M_{\odot}$ will be considered, so that a large fraction of the mass must be thrown off in the instability process in order that the mass of the remaining nucleus be reduced to a value comparable with Chandrasekhar's limit.

When the star becomes rotationally unstable the material thrown off expands against the powerful gravitational field of the star. At first sight the expansion seems to be exactly the reverse of the compression that occurred during the contraction of the star, but it turns out that there is an important difference between these two processes. If we consider the stellar material during the stage of compression *before* the break-down point is reached, and we imagine that some cause momentarily reduces the temperature sufficiently for nuclear reactions effectively to cease, then absorption of energy by the nuclear reactions will also effectively cease. Thus a surplus of energy will be available that raises the temperature of the material until statistical equilibrium is again set up. This means that before the break-down point is reached the material *during compression* is automatically governed to remain in statistical equilibrium. An opposite situation occurs during the expansion of the material thrown off by rotational instability. For if nuclear reactions cease during expansion then the supply of energy *decreases* since energy is no longer made available by the nuclear reactions. Accordingly there is no automatic governing effect during expansion and material once frozen will become even more effectively frozen as the expansion proceeds.

In view of the remarks of the previous paragraph it is evidently plausible to suppose that some of the ejected material remains frozen during the expansion, and in the following discussion it will be assumed that the quantity of this frozen material amounts to a few per cent. of the total material thrown off by the instability process. Thus the present discussion divides into two parts:—

Case 1.—To obtain the composition of material that moves during the expansion in a reverse direction along a track in the T, ρ'-diagram similar to that shown in Fig. 1.

Case 2.—To obtain the composition of material that remains frozen during the instability process.

The quantity of material satisfying Case 2 is only a few per cent. of the material satisfying Case 1. Case 1 is considered in 8.2 and Case 2 in 8.3.

8.2. *The composition of material in Case* 1.—The final composition of this material is determined by the composition at the time it crosses the line AB, since the material must freeze when T decreases below 4×10^9 °C. Thus the final

* F. Hoyle. In course of publication.

composition is close to that occurring at the point D in the T, ρ'-diagram. It will be shown that the composition at this point is roughly uniform over the lower half of the periodic table.

The relative abundance $n_{A'}^{Z'}/n_A^Z$ is easily obtained from equation (19) in terms of y, T. Thus we have

$$\log n_A^Z = 34\cdot08 + \frac{3}{2}\log T + \frac{3}{2}\log A + \frac{A}{A'}\left(\log n_{A'}^{Z'} - 34\cdot08 - \frac{3}{2}\log T - \frac{3}{2}\log A'\right) + \frac{4\cdot73A}{T}\left(\frac{Q_A^Z}{A} - \frac{Q_{A'}^{Z'}}{A'} + 0\cdot543y\left\{\frac{Z'}{A'} - \frac{Z}{A}\right\}\right). \tag{50}$$

On the line AB of Fig. 1 we have $T=4$, and to a sufficient approximation the experimental values of Q_A^Z/A can be used. In addition the term in y can be neglected when ρ' is of order 10^7 gm. per cm.^{3}. To this approximation equation (50) does not involve ρ', and if $n_{A'}^{Z'}$ is specified for a given pair of values A', Z' the corresponding values of n_A^Z can be worked out for all other A, Z. The density ρ is then given by

$$\rho \simeq \rho' \simeq m_P \sum_{A,Z} A n_A^Z,$$

since the contribution of the free neutrons to the density of the material is small when $T=4$, $\rho'=10^7$ gm. per cm.^{3}. Thus specifying $n_{A'}^{Z'}$ is equivalent to specifying ρ'. In fact it can be shown by calculation that if we specify n_{56}^{26} for example, then ρ' is roughly proportional to n_{56}^{26} for values of ρ' lying between A and B in Fig. 1. The value $n_{56}^{26}=10^{28}$ was adopted in calculating the following selection of values of n_A^Z (it being more convenient to specify $n_{A'}^{Z'}$ than to specify ρ):—

Table XIV

Element	He^4	O^{16}	Si^{28}	Fe^{56}	Cu^{63}	Kr^{82}	Sn^{118}	Pb^{208}
$\log n_A^Z$	27·2	20·2	23·7	28	26·7	18·8	−3·3	−222·8

The total density given by this distribution is close to 10^7 gm. per cm.^{3}. Thus the distribution at the point D in Fig. 1 has approximately the form given in Table XIV.

The rather small oxygen abundance given in Table XIV arises from adopting an approximation in which the whole of the material freezes at $T=4$. In a more accurate treatment it would be necessary to take account of the fact that nuclear reactions among light elements will persist to a lower temperature than for nuclei with $A>30$. This must be the case because the proton and α-particle energies required to penetrate the Coulomb barriers is less for the light nuclei. Thus in the cooling of the material a temperature will be reached at which nuclear reactions can be regarded as taking place for $A<30$, but as being effectively frozen for $A>30$. When this occurs the large helium content shown in Table XIV tends to be converted into elements with atomic weights up to 30. This effect can be seen by considering the helium-oxygen ratio at temperatures less than $T=4$. From equation (50) we then have

$$\log n_{16}^8 = 34\cdot08 + \frac{3}{2}\log T + \frac{3}{2}\log 16 + 4(\log n_4^2 - 34\cdot08 - \frac{3}{2}\log 4\mathrm{T}) + \frac{4\cdot73}{T}(Q_{16}^8 - 4Q_4^2). \tag{51}$$

Putting $Q_{16}^8/16=8\cdot54$, $Q_4^2/4=7\cdot57$ it is easy to see from (51) that n_{16}^8 becomes comparable in magnitude with n_4^2 when T is close to 3. For this temperature it can be shown that

$$\log n_N \simeq \log n_P \simeq 21.$$

It may be concluded that if equilibrium among light nuclei persists down to $T=3$ then the helium present at $T=4$ will be largely converted into elements lying between carbon and silicon.

8.3. *The composition of material in Case 2.*—The results given in Table XI show that the composition of material that remains frozen during the expansion depends on the density attained at the onset of the instability process. The most abundant element present in a particular sample of this material must consequently vary according to the region of the star that supplies the material in question. Thus, if for the sake of definiteness, the central density is taken to be 5×10^{11} gm. per cm.^{3} then material derived from near the centre of the star must have $A_{\max}$ close to 240, while on the other hand material derived from the outer parts of the star will only attain a density comparable with the mean density of the star. Allowing for the central density exceeding the mean by a factor of 20, it follows that the latter material attains a density of order $2\cdot5 \times 10^{10}$ gm. per cm.^{3}, and according to Table XI such material must have $A_{\max} < 110$. It is to be expected therefore that the average composition of all material satisfying Case 2 will consist of a fairly uniform distribution of the elements over the upper half of the periodic table, since such an average must include contributions from all parts of the star.

It is tacitly assumed in the previous paragraph that material can attain a density of 5×10^{11} gm. per cm.^{3} without the statistical break-down point being reached. The values given in Table XIII show that this will be the case if the parameter α is greater than 5.

The discussion of the composition in the case where material reaches the break-down point is more complicated than that given above, since it is then necessary to replace the statistical equations by a consideration of individual reactions. This discussion will not be attempted in the present paper. It may however be noted that the slowing down of nuclear reactions at the break-down point must make such material particularly likely to remain frozen during the subsequent expansion.

9. *The Effect of β-processes on the Rate of Collapse of a Star*

9.1. *The Rate of Collapse of the Star.*—It is convenient at this stage to discuss the condition (iv) of Section 5. The requirement is that a sufficient number of interchanges between neutrons and protons must take place for equation (18) to be applicable.

If the time required for (18) to be established is less than the time of free fall of the star, then the star will collapse freely along the portion FG of the track of stellar material in the ρ', T-diagram. On the other hand, if the time of free fall is less than the time required for (18) to be established then the evolution of the stellar material along FG is determined by the time required for (18) to be established and not by the time of free fall. The reason for this is clear, for if equation (18) is not satisfied, free neutrons will not be produced at the rate calculated above. But it is the production of free neutrons that removes energy and is responsible for the rapid collapse of the star. This means that free fall of the star is not possible unless equation (18) is satisfied. The situation is that the star will collapse at a rate given either by the time required for condition (iv) to be established or by the time of free fall, according to which of these two times is the longer. The purpose of the present section is to estimate these two times.

The time required for the free fall of the star is easily estimated, so far as an order of magnitude is concerned, from the formula $\sqrt{(R^3/GM)}$ where R is the radius of the star at F. This formula can be written as $\sqrt{(3/4\pi G\bar{\rho})}$ where $\bar{\rho}$ is the mean density of the star at the time when the material near the centre is at F. This requires a central density of about 2×10^8 gm. per cm.^{3} which corresponds to a mean density of about 10^7 gm. per cm.^{3}. The value of $\sqrt{(3/4\pi G\bar{\rho})}$ for $\bar{\rho}=10^7$ gm. per cm.^{3} is 0·42 sec., which may be compared with the result obtained below, that the time required for (18) to be established is of the order of 100 sec. It is seen therefore that the rate of collapse is determined by the time required for equilibrium to be set up between the protons and neutrons.

9.2. *Concerning β-processes.*—It is seen from Table XI that $(2Z/A_{\max})$ is close to 0·5 when ρ' exceeds 10^{11} gm. per cm.^{3}. This value is much less than the value of $2Z/A$ occurring under laboratory conditions. Thus even for uranium the quantity $2Z/A$ is as large as 0·77 while for lighter elements $2Z/A$ is of course greater still. It follows therefore that the nuclei considered in the discussion of Section 7 would be extremely unstable against β-disintegration under laboratory conditions. Thus such nuclei thrown off by a rotationally unstable star must rapidly be converted into nuclei with the values of $2Z/A$ found on Earth. An estimate for the period of decay under laboratory conditions can easily be given. The upper limit of the β-spectrum is given by putting $2Z/A=0{\cdot}5$ in $(Q_A^Z-Q_A^{Z-1})$. The value of $(Q_A^Z-Q_A^{Z-1})$ is given to sufficient accuracy by differentiating Q_A^Z with respect to Z. This gives

$$84\left(1-\frac{2Z}{A}\right)-\frac{1{\cdot}25Z}{A^{\frac{1}{3}}}\text{ m.m.U.,} \tag{52}$$

and on putting $2Z/A=0{\cdot}5$ we obtain an upper limit to the β-spectrum of $(42-0{\cdot}312/A^{\frac{1}{3}})$ m.m.U., which is close to 39 M.V. Now according to the Fermi theory of β-decay the lifetime of an allowed transition is given by

$$\frac{2^5\pi^4g^2W_0^5}{15h^7c^6}, \tag{53}$$

when the following assumptions are made:—

(i) The upper limit W_0 of the spectrum is appreciably greater than the rest mass of the electron.

(ii) The effect of the Coulomb field of the nucleus on the wave function of the emitted electron is neglected.

(iii) The matrix element for the β-transition is put equal to unity, so that g is the β decay constant.

The requirement (i) is satisfied by a large margin in the present problem, but (ii) and (iii) are not satisfied for nuclei at the top end of the periodic table. However, it is known that the effect of the Coulomb field is to decrease the lifetime of an unstable nucleus while on the other hand the matrix element contributed by the initial and final wave functions of the nucleus increases the lifetime as the atomic weight A increases. The laboratory data shows that these two effects approximately cancel each other, so that (53) may be used as an approximate formula over the whole range of atomic weight.

The constant g is not known from theoretical considerations and the usual procedure is to fix g from the observed lifetime of a typical β-decaying substance. As a reasonable estimate a lifetime of one second for an allowed β-reaction with a spectrum of upper limit 7·5 M.V. may be taken. Thus the case discussed above in which the upper limit is 39 M.V. would have a lifetime of $2\cdot6\times10^{-4}$ sec. It is particularly to be noted that this lifetime refers to laboratory conditions and could not arise for material in a star. The reason for this is that the emission of electrons is prevented by degeneracy. Thus if we take a nucleus with a value of $Z>Z_{\max}$, electrons cannot be emitted because the effect of degeneracy is to fill the electron levels to a value *higher than the upper limits of the β-spectra*. When $Z=Z_{\max}$ the upper limit of the β-spectrum becomes equal to the top of the energy levels filled by the electrons while for $Z>Z_{\max}$ the β-spectra have upper limits that exceed the highest energies of the degenerate electrons. This situation was of course taken care of in Section 7 when the maximum value of $(Q_A^Z-0\cdot543yZ)$ with respect to Z was obtained.*

It follows from the remarks of the previous paragraphs that under equilibrium conditions β-disintegrations in a star only occur for nuclei with $Z<Z_{\max}$. Thus in order to obtain the rate at which β-disintegrations occur under statistical equilibrium it is necessary to estimate the ratio $n_A^Z/n_A^{Z_{\max}}$ for $Z<Z_{\max}$. This ratio is obtained immediately by putting $A'=A$, $Z'=Z_{\max}$ in equation (50), which gives

$$\log\left(\frac{n_A^Z}{n_A^{Z_{\max}}}\right)=-\frac{4\cdot73}{T}\{(Q_A^{Z_{\max}}-0\cdot543yZ_{\max})-(Q_A^Z-0\cdot543yZ)\},\tag{54}$$

Expanding the right-hand side of (54) by Taylor's theorem gives

$$\log\left(\frac{n_A^Z}{n_A^{Z_{\max}}}\right)=-\frac{2\cdot36}{T}\cdot(Z_{\max}-Z)^2\left\{\frac{d^2}{dZ^2}(Q_A^Z-0\cdot543yZ)\right\}_{Z=Z_{\max}},$$

$$\text{when } 0<(1-Z/Z_{\max})^2\ll 1,\ \text{since}\left\{\frac{d}{dZ}(Q_A^Z-0\cdot543yZ)\right\}_{Z=Z_{\max}}=0.\tag{55}$$

Equation (55) gives the required formula for the number of nuclei that are unstable against β-decay.

Now the expression (53) can no longer be used on account of electron degeneracy. For a nucleus of charge Z such that $0<1-Z/Z_{\max}\ll 1$ the difference between the upper limit W_0 of the β-spectrum and the highest energies of the degenerate electrons is given by

$$\frac{(Z-Z_{\max})^2}{2}\left\{\frac{d^2}{dZ^2}(Q_A^Z-0\cdot543yZ)\right\}_{Z=Z_{\max}}=\frac{84(Z-Z_{\max})^2}{A}\Big\}.\tag{56}$$

Thus β-disintegration can only occur if the energy W of the emitted electron satisfies the inequality

$$W_0>W>W_0-\frac{84(Z-Z_{\max})^2}{A},\tag{57}$$

* At first sight it seems strange that the mathematics deals so easily with this matter. But this is the outstanding feature of statistical mechanics. So long as statistical equilibrium occurs the equilibrium properties of an assembly can be determined without direct reference to individual processes being necessary.

where W, W_0 are measured in milli-mass units. The fraction of the electrons emitted in this range is given by

$$\int_{W_0-84(Z-Z_{\max})^2/A}^{W_0} W(W^2/m_e^2c^4-1)^{\frac{1}{2}}(W_0-W)^2dW \div \int_0^{W_0} W(W^2/m_e^2c^4-1)^{\frac{1}{2}}(W_0-W)^2dW, \tag{58}$$

where m_ec^2 is the rest mass of the electron, measured in m.m.U. The result (58) follows immediately from the well-known spectrum

$$W(W^2/m_e^2c^4-1)^{\frac{1}{2}}(W_0-W)^2dW$$

given by the Fermi theory. When $W_0/m_ec^2 \gg 1$ and $84(Z-Z_{\max})^2/A \ll W_0$, the expression (58) is given to sufficient accuracy by

$$10\left\{\frac{84(Z-Z_{\max})^2}{W_0A}\right\}^3. \tag{59}$$

Thus the lifetime against β-disintegration in a star is given by dividing (53) by (59). Using the value $2\cdot6\times10^{-4}$ sec. obtained above for (53) this gives a lifetime of

$$2\cdot6\times10^{-5}\left\{\frac{W_0A}{84(Z-Z_{\max})^2}\right\}^3 \text{sec.}$$

Since it is sufficiently accurate to put $W_0=42$ m.m.U. for all $Z<Z_{\max}$ and

$$\frac{n_A^Z}{n_A^{Z_{\max}}}=10^{-\frac{4\cdot73}{T}\cdot\frac{84(Z-Z_{\max})^2}{A}}$$

it follows that the number of β-disintegrations per sec. given by all nuclei of atomic weight A is

$$\frac{3\cdot1\times10^5n_A^{Z_{\max}}}{A^3}\sum_{Z<Z_{\max}}(Z-Z_{\max})^6 10^{-\frac{4\cdot73}{T}\cdot\frac{84(Z-Z_{\max})^2}{A}} \tag{60}$$

and the *total* number of β-disintegrations occurring per cm.^{3} per sec. is

$$3\cdot1\times10^5\sum_A\frac{n_A^{Z_{\max}}}{A^3}\sum_{Z<Z_{\max}}(Z-Z_{\max})^6 10^{-\frac{4\cdot73}{T}\cdot\frac{84(Z-Z_{\max})^2}{A}} \tag{61}$$

The main contribution to the series in (60) comes from Z close to $Z_{\max}$ for all values of A. In order to evaluate (60) for various A it is necessary to specify $4\cdot73/T$. Over the portion of the T, ρ'-diagram that is of interest (that is, between F and G) we may take 0·5 as a representative value for $4\cdot73/T$. The following table gives expression (60) divided by $n_A^{Z_{\max}}$, for a set of values of A:—

Table XV

A	16	56	126	252
Expression (60)/$n_A^{Z_{\max}}$	0·18	0·43	0·65	0·92

Table XV shows the surprising result that the number of β-disintegrations is a roughly constant multiple of $n_A^{Z_{\max}}$ for all values of A from 56 up to the top of the periodic table. It follows therefore that (61) is given approximately by

$$\frac{2}{3}\sum_A n_A^{Z_{\max}} \simeq \frac{2\rho'}{3m_PA_{\max}}.$$

Thus we reach the important conclusion that the number of β-disintegrations occurring per cm.^{3} per sec. for material in the heavy element zone is of order $2\rho'/3m_P A_{\max}$ no matter what value of ρ' is chosen on the portion F to G of the track of the stellar material.

Under equilibrium conditions the number of inverse β-processes involving electron capture must be equal to the number of direct disintegrations calculated above. Thus the number of inverse β-process is of order $2\rho'/3m_P A_{\max}$ per cm.^{3} per sec. A direct calculation of the number of inverse β-processes could be given but it is more convenient to follow the method given above. It may be noted that the inverse β-processes arise from electron capture by nuclei with $Z>Z_{\max}$. The reason for this is that the energy of the captured electrons must exceed the upper limits of the β-spectra and this requires $Z>Z_{\max}$. This result deserves comment for it means that the direct β-disintegrations occur for $Z<Z_{\max}$ while the inverse processes occur for $Z>Z_{\max}$. Thus there can be no detailed balancing. The *total* number of direct β-disintegrations in an equilibrium state is of course equal to the number of inverse processes even if there is no detailed balancing in individual reactions. At first sight this lack of detailed balancing seems to be in opposition to the well-known argument of statistical mechanics that seeks to establish detailed balancing from a consideration of the Hermitian character of the Hamiltonian of the assembly. But there is no such contradiction because the argument based on the Hermitian property of the Hamiltonian depends on the assembly being in strict thermodynamic equilibrium, whereas our assembly is not in thermodynamic equilibrium because the density of free neutrinos is effectively zero (the system is in equilibrium in the sense that average values can be calculated for n_A^Z, n_P, n_N and n_ϵ, but this does not constitute strict thermodynamic equilibrium and is really a steady state distribution). The breakdown of thermodynamic equilibrium is seen as soon as we consider reactions involving neutrinos. Thus a neutrino is always *emitted* in a state of positive energy, but a neutrino can never be *captured* in a state of positive energy. Accordingly the neutrino reactions are not thermodynamically reversible and it is therefore not surprising that detailed balancing does not occur in these reactions.* It is precisely this property of the neutrino reactions that requires the direct β-disintegrations to occur for $Z<Z_{\max}$ and the inverse reactions to occur for $Z>Z_{\max}$.

We are now in a position to estimate the length of time required for statistical equilibrium between neutrons and protons to be set up. The way in which statistical equilibrium comes to be set up is easily understood. Consider a departure from statistical equilibrium in which

$$\left.\begin{aligned} n_A^Z &> (n_A^Z)_{\text{equilibrium}}, \quad \text{when} \quad Z>Z_{\max},\\ n_A^Z &< (n_A^Z)_{\text{equilibrium}}, \quad \text{when} \quad Z<Z_{\max},\\ \Sigma n_A^Z &= \sum_A (n_A^Z)_{\text{equilibrium}}, \end{aligned}\right\} \tag{62}$$

This has the effect of reducing the number of direct β-disintegrations below the equilibrium value, while the rate of the inverse β-processes on the other hand

* Gamow and Schoenberg, *Phys. Rev.*, **59**, 539, 1941, use detailed balancing in neutrino reactions in order to calculate n_A^{Z-1}/n_A^Z. The writer is not in agreement with this assumption as the value of n_A^{Z-1}/n_A^Z must be calculated from the full scheme of equations involving equilibrium between the nuclei and cannot be calculated from the neutrino reactions alone.

is increased. This follows immediately since the direct disintegrations occur for nuclei with $Z<Z_{max}$ and the inverse processes for $Z>Z_{max}$. Thus under the conditions (62) the rate of the inverse processes must exceed the rate of the direct disintegrations. Moreover the rate of the direct processes can be reduced to a negligible value by putting $\sum_{Z<Z_{max}} n_A^Z \simeq 0$. The rate of the inverse processes on the other hand cannot be much increased above the equilibrium value. Accordingly it follows that an excess of inverse processes over direct disintegrations amounting to $1/2\ m_P$ per gm. can be produced in a time of order

$$\frac{1}{2m_P} \cdot \frac{2}{3A_{max}m_P} = \frac{3A_{max}}{4} \text{ sec.},$$

which is of the order of 100 sec.

The above investigation shows that the time taken for β-processes to establish statistical equilibrium between protons and neutrons is appreciably longer than the time of free fall of the star which is of order one second (see 9.1). Thus the time scale of the contraction is determined by the β-processes and not by the time of free fall. This result enables the loss of energy by neutrino emission to be calculated fairly accurately, since the total number of β-processes occurring before the onset of rotational instability cannot appreciably exceed $1/2m_P$ per gm. Now the great majority of these reactions occur for nuclei with Z close to Z_{max} and consequently the energy of emission of the neutrino must be small compared with the upper limits of the β-spectra. Expressed quantitatively the energy of an emitted neutrino must be less than

$$\frac{84(Z_{max}-Z)^2}{A} \text{ m.m.U.}$$

Thus it follows, since $A_{max}>50$, that 1 M.V. is a representative estimate for the average energy of the emitted neutrinos. Accordingly the energy lost by neutrino emission cannot be much greater than 5×10^{17} ergs per gram of stellar material. This value may be compared with the energy required to produce the large number of free neutrons present in material with density greater than 10^{11} gm. per cm.^{3}. The energy required to liberate these neutrons from the nuclei is approximately $c^2/135$ per gm. Accordingly the energy required to liberate the neutrons is about 7×10^{18} ergs per gm. as against 5×10^{17} ergs per gm. for the loss of energy by neutrino emission. This result shows that the energy lost in neutrino emission is small compared with the energy absorbed by the other nuclear reactions. This provides justification for our neglect of the neutrino energy loss in the work of Sections 6 and 7.

Finally the important point may be noted that because the time-scale for the contraction of the star is determined by the time required for equilibrium to be set up in the β-processes, the material of the star is never strictly in a steady equilibrium state. In a fuller discussion it would be necessary to take account of variations from the steady state and this would undoubtedly involve a more complicated theory than that given above. This has not been attempted in the present paper because it is desirable to begin by exploring the equilibrium theory. Moreover the present method of determining the track of stellar material in the T, ρ'-diagram already takes account of the main features associated with the changing state of the material.

10. *Novae, Supernovae and the Origin of the Solar System*

The work of previous sections shows that when material near the centre of a collapsing star reaches the point E of Fig. 1 the hydrostatic equation of support ceases to hold even as an approximation. Thus the collapse and the onset of rotational instability is catastrophic *provided rotational instability does not occur before the point E is reached.* Collapsing stars that satisfy the latter condition will be regarded as supernovae.*

According to this theory the outburst of a supernova is even more rapid than is suggested by observation, which gives a time of about two days for the rapid rise to maximum light. The explanation of the slower rise of the observed light is clear, for the supernova will not emit appreciable energy in the visible part of the spectrum until the material thrown off has expanded to an envelope of dimensions large compared with the radius of the star. As the envelope expands high frequency radiation is degenerated and the proportion of light emitted in the visible spectrum increases. Thus the time taken for the light emitted in the visible spectrum to reach its maximum is associated with the time required for the expansion of the envelope and is not directly connected with the time-scale of the instability process (provided this time-scale is less than the time required for the expansion of the envelope).

If a star becomes rotationally unstable before the point E in Fig. 1 is reached then the equation of support (1) will be satisfied to a good approximation throughout the collapse. This means that the state of rotational instability will be approached gradually over a time that is of the order of, or greater than, a thousand years. It seems likely that instability in this case will take the form of a series of minor explosions rather than the star remaining completely stable until a state of such violent instability is reached that an outburst of the supernova type occurs. That is, the star will splutter as it slowly contracts. Thus the point E divides collapsing stars into two groups. The first group of stars reach E and undergo a catastrophic outburst, whilst stars not reaching E undergo a series of minor explosions similar in character to the outbursts of an ordinary nova.

The existence of the point E in the T, ρ'-diagram provides a natural theory of the difference between novae and supernovae. Moreover in this theory there is no invention of an unspecified physical process in order to produce the energy required for the material to be thrown off, since this energy is derived from the gravitational potential energy of the star. In comparing the theory with observation it is therefore necessary that the magnitude of the gravitational potential energy of the remaining stellar nucleus after outburst shall be greater than the kinetic energy at infinity of the material thrown off. If $M\mu_e^2/5{\cdot}75M_\odot \gg 1$ the material thrown off has a mass comparable with M, and the remaining stellar nucleus has a mass comparable with Chandrasekhar's limit $5{\cdot}75M_\odot/\mu_e^2 = M_{cr}$. Thus if v is the velocity at infinity of the ejected material it is necessary that $Mv^2/2$ shall be of the same order but less than GM_{cr}^2/R. It is of interest to apply this condition in the case of the supernova of 1054 A.D. The observational data lead to the values $M \simeq 15M_\odot$, $v \simeq 10^8$ cm. per sec., $R = 10^9$ cm. Thus since $5{\cdot}75M_\odot/\mu_e^2 \simeq 1{\cdot}5M_\odot$, it is seen that $Mv^2/2 \simeq 1{\cdot}5 \times 10^{50}$ ergs while $GM_{cr}^2/R \simeq 6 \times 10^{50}$ ergs. Accordingly the supernova of 1054 A.D. satisfies the requirements of the theory of rotational instability.

* This may be compared with the Gamow-Schoenberg neutrino emission theory.

The distribution of the elements in the solar system raises a question of interest in relation to the present work. It is easy to see that the theory of the synthesis of the elements given in Section 8, taken together with the theory of the origin of the solar system recently proposed* by the writer, provides a satisfactory explanation of this distribution so far as elements heavier than helium are concerned. It is important to notice that this original planetary material contains no hydrogen. The present hydrogen content of the solar system is regarded as due to the sweeping up of interstellar hydrogen by the envelope of planetary material. In order to consider this process quantitatively let s be the radius of the original envelope of planetary material, t the time required for the condensation of the gaseous material into solid bodies, u the velocity of the Sun relative to the interstellar hydrogen, and σ the density of the interstellar hydrogen. Then the amount of hydrogen acquired by the planetary system is of order

$$\pi s^2 \sigma u t.$$

A recent investigation by the writer suggests that t must be of order 3×10^8 years. With this value for t, and $s = 3 \times 10^{14}$ cm., $\sigma = 10^{-22}$ gm. per cm.^{3}, $u = 2 \times 10^6$ cm. per sec., we obtain a mass of hydrogen amounting to about twenty per cent. of the total mass of the planets. This is of the required order of magnitude.

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Cambridge :
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* F. Hoyle, *Proc. Camb. phil. Soc.*, **40**, 256, 1944; *M.N.*, **105**, 175, 1945.